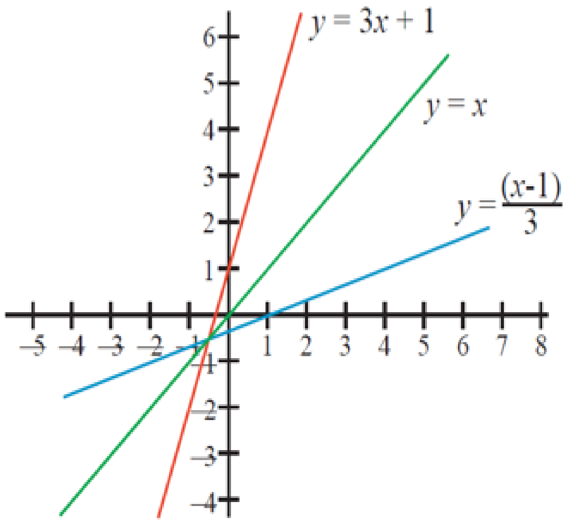

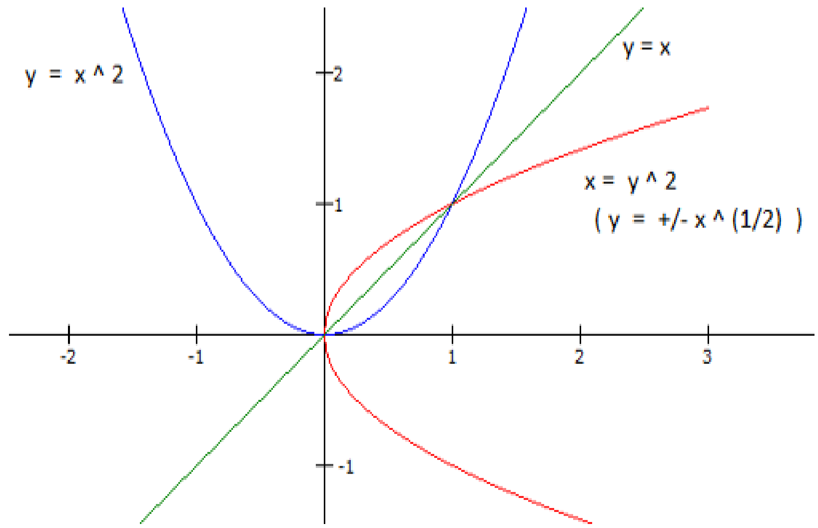

Composition of Inverse Functions

by Dan Umbarger

www.mathlogarithms.com
www.calculusbook.net
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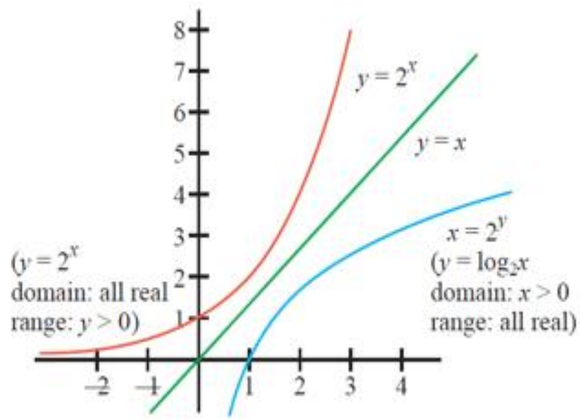
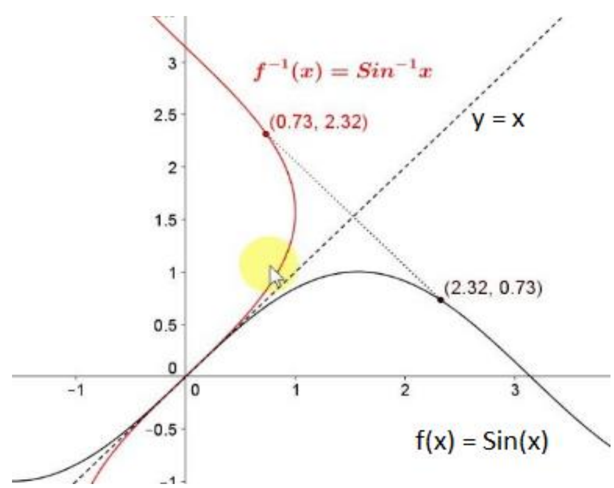




Composition of Inverse Functions

The biggest reason that many people have difficulty understanding mathematics is that the ability to understand new ideas is often predicated or built upon the understanding of earlier ideas or concepts which they have not learned. Those foundation ideas may not have been taught, they may have been taught poorly, or they may have been taught but were forgotten. Two really good examples are the identities

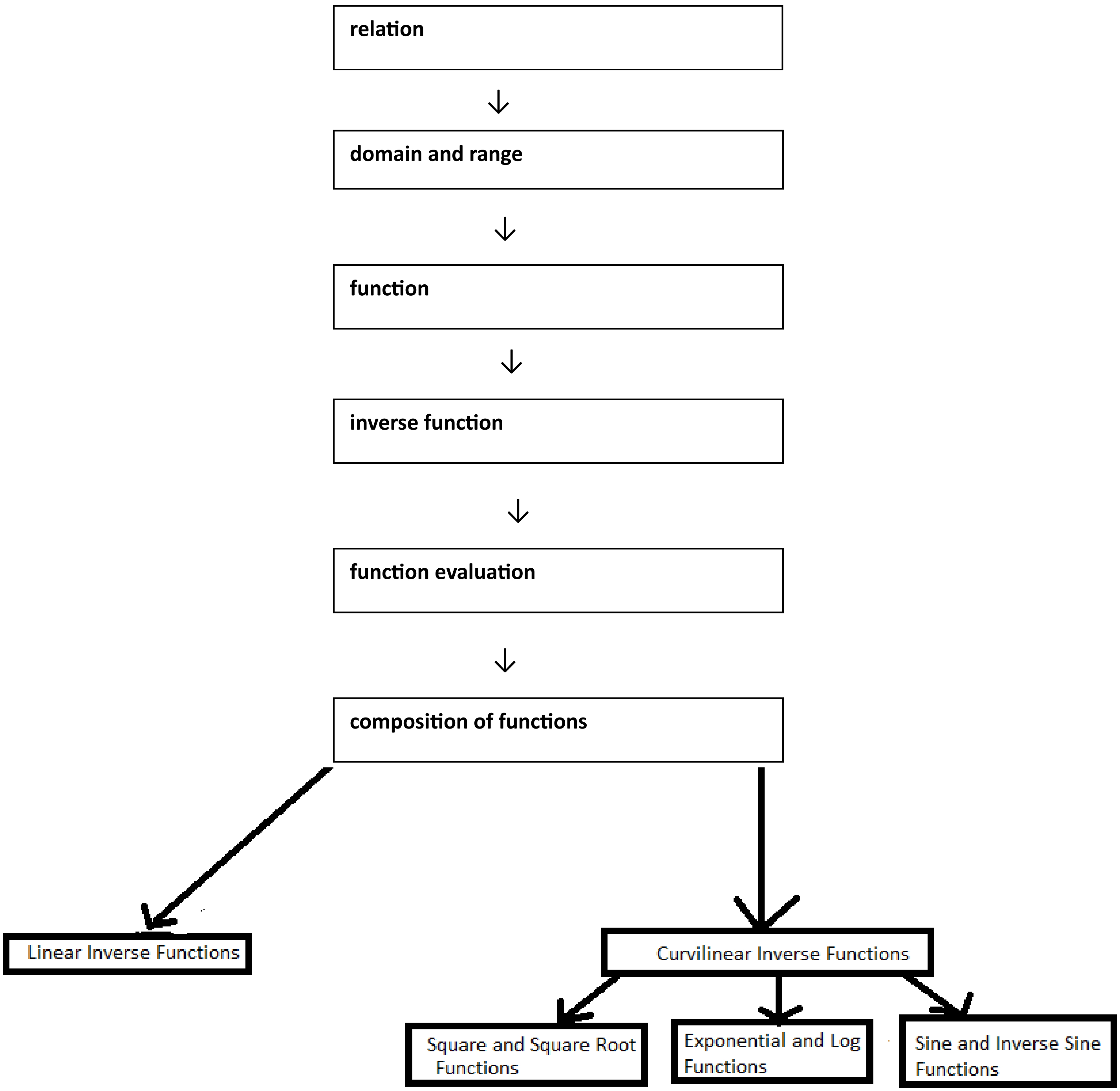
$log_b b^x$ = **x** and $b^{log_b x}$ = **x.**

For some students these identities seem to just jump out of nowhere (deus ex machina) when solving equations in upper level math. The student can learn to apply them and "get the correct answer" but if they do not understand why these identities work then they are learning "mathemagic" and not "mathematics". The result is that many students are being set up for eventual bewilderment, frustration, and confusion. Teachers of classes that apply these identities in the process of "getting an answer" are so smart that they do not understand how important the understanding of them is to the long term growth and comfort level of the student. To be fair upper level math teachers are required to teach a lot of material in a fixed amount of time and in order to do so they must make make assumptions about what the students already know. In theory the two identities shown above should have been taught down in Pre-Calculus somewhere but there are a lot of high schools who are unable to recruit and maintain qualified math teachers. As a result these two identities are taught as magic formulas with no justification or explanation. This is especially the case for small rural schools with small student populations or large inner city schools with rapid turnover of teachers. The point here is that many students end up in advanced math classes using "magic formulas and identities" that they do not understand. Also, it should be said that many students have a difficult time learning information that has no immediate application and the identities shown above may not be used in application situations until later in the math or science curriculum.

It turns out that these two identities fall into a category of equations called "composition of inverse functions" and the justification for them is the same as for other concepts and and equations which have been used since Algebra I. If a student can connect new ideas with previous ideas it makes new learning much more comfortable.

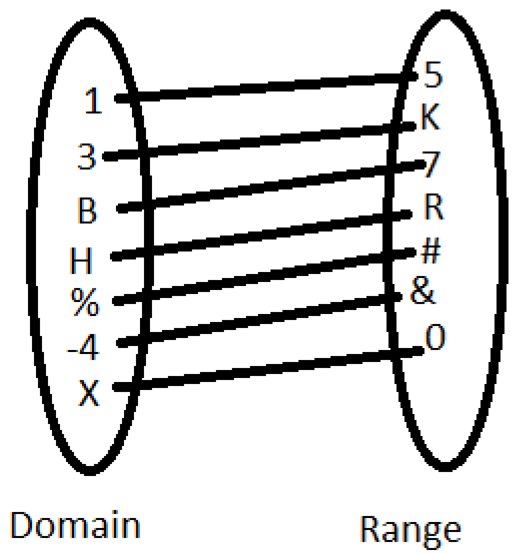
The following ideas all synergize to explain "composition of inverse functions": relation, domain and range, function, inverse function, function evaluation, composition of functions, types of inverse functions. If a student has not mastered all these ideas then he/she will not be able to understand "composition of inverse functions."

Pictures are a quick way to communicate information. Compare the previous text: "The following ideas all synergize to explain composition of functions: relation, domain and range, function, inverse function, function evaluation, and composition of functions" with the following



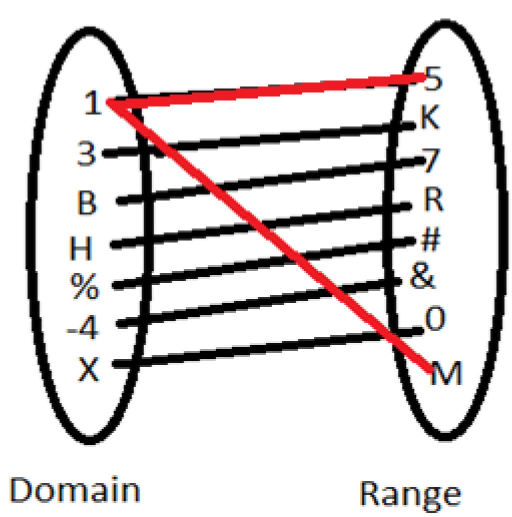
Relation, Domain, Range:

A relation is a set of ordered pairs: { (1,5) , (3,K), (B,7), (H, R), (%,#), (-4,&), (X, 0) }
All the first values in a relation (1, 3, B, H, %, -4, X above) are called domain values.
All the second values in a relation are called range values. Relations are often shown pictorially:

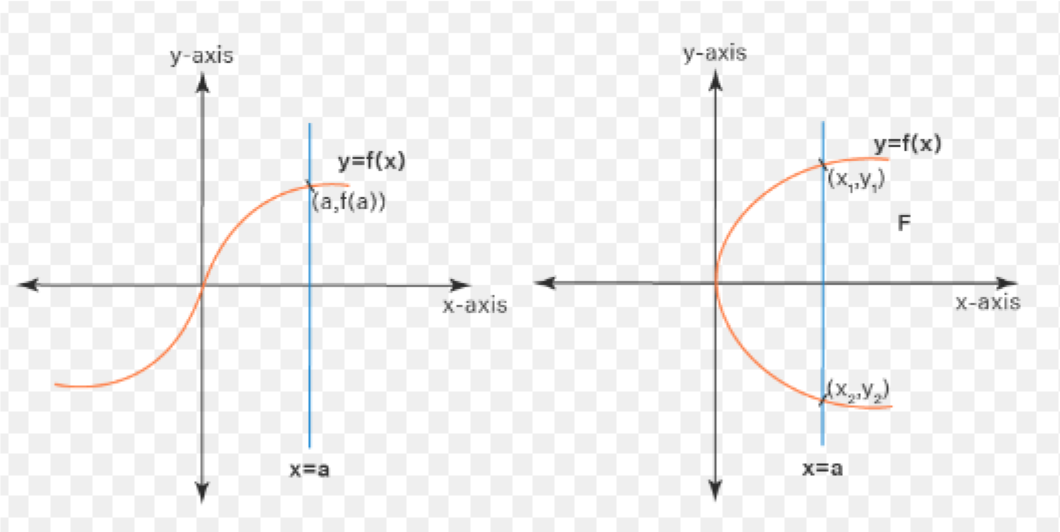


Function (definition #1)

A function is a type of relation with the requirement that each member of the domain can only be connected to one member of the range. The relation shown above is a function. The relation shown below is not a function {(1,5), (3,K), (B,7), (H, R), (%,#), (4,&), (X,0), and (1,M) } because the number 1 in the domain is paired with more than one member in the range.



Another way to distinguish functions from relations is called the "vertical line test". This is done by showing a graph (which is a set of infinite ordered pairs) with a vertical line. If a vertical line only passes through 0 or 1 points in a graph that graph is a function.



Vertical Line Test- ✓
(It is a function)

Vertical Line Test- X
(Not a function)

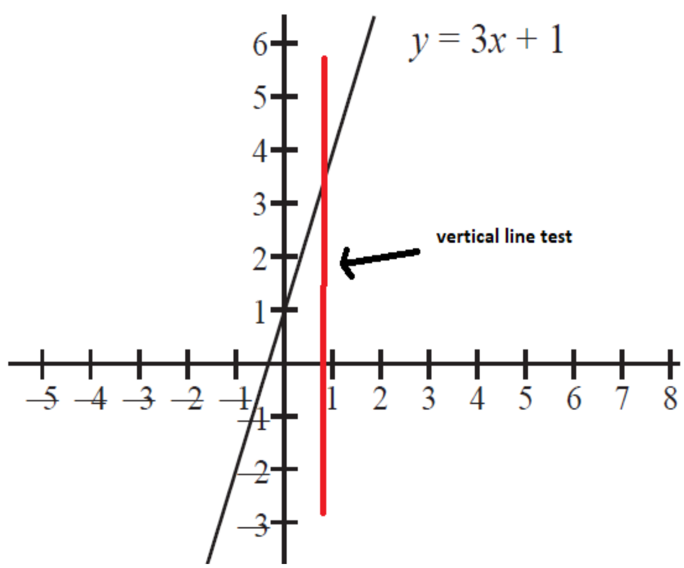
Function (Definition #2)

A function is a relation **together with a rule** which maps each member of the relation's domain to at most 0 or 1 member of the range.

Rule

$$\boxed{y = 3x + 1}$$

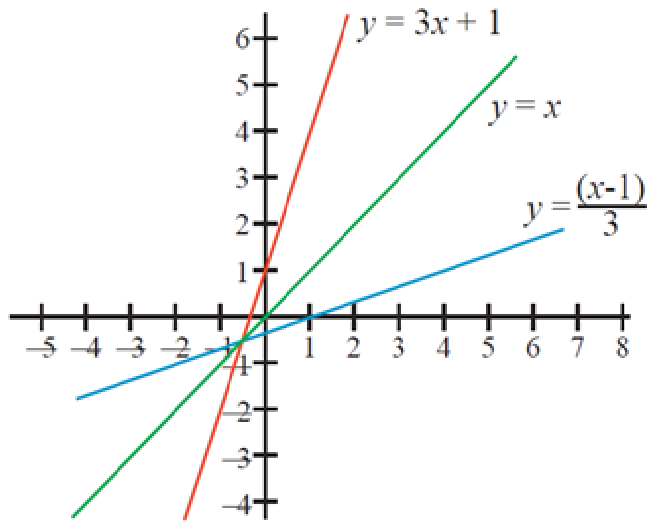
x (domain)	y (range)
–2	–5
–1	–2
0	1
1	4
2	7



Inverse Function

An inverse function (if it exists) can be found by exchanging the x and y variables in the given function. For y = 3x + 1 we get x = 3y + 1. We then traditionally solve this new equation for y… $y = \frac{x-1}{3}$. There is an interesting geometric relation between the graph of a function and that of its inverse. Both graphs are symmetric around the line y = x. If you fold the graph along the line y = x the graph of both functions match up.

e.g.,	Original function	y	$=$	$3x + 1$
	Inverse function	x	$=$	$3y + 1$
	or	$x - 1$	$=$	$3y$
	or	$3y$	$=$	$x - 1$
	or	y	$=$	$(x - 1)/3$



y = 3x + 1 x = 3y + 1 ($y = \frac{x-1}{3}$)

x (domain)	y (range)		x (domain)	y (range)
–2	–5	⇐compare⇒	–5	–2
–1	–2	⇐compare⇒	–2	–1
0	1	⇐compare⇒	1	0
1	4	⇐compare⇒	4	1
2	7	⇐compare⇒	7	2

This should not be surprising. The inverse function was formed by exchanging the x and the y in the original function. This is what causes the two graphs to be symmetric around the line $y = x$.

Function Evaluation

Function evaluation is done by substituting values into the domain of the function rule.

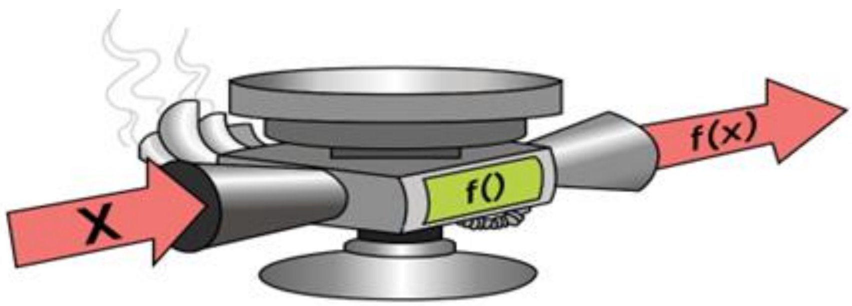
Rule

$$\boxed{y = 3x + 1}$$

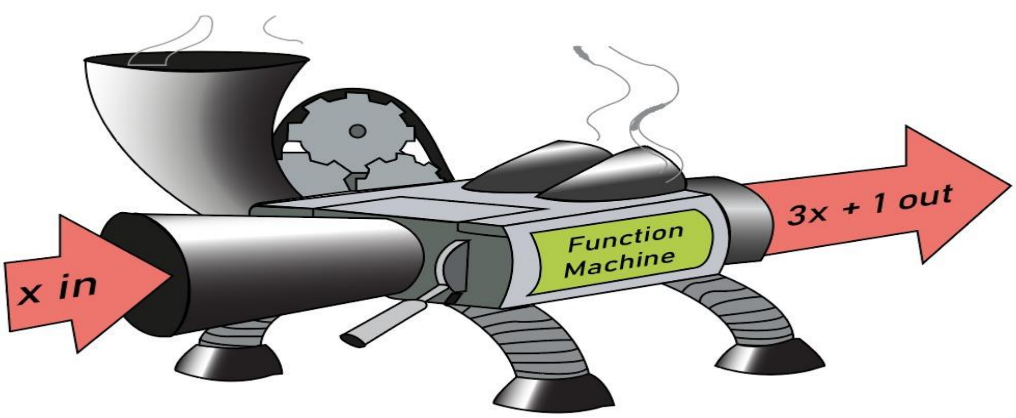
x (domain)	y (range)
−2	−5
−1	−2
0	1
1	4
2	7

For beginners function machine cartoons are a good way to introduce function evaluations.

y = f(x)



f(x) = 3x + 1



Composition of Functions

Composition of functions occurs when the result of one function is used as input to another.

e.g., $f(x) = 2x + 1$ $\quad g(x) = 3x - 1$

x	$f(x)$	$g(f(x))$	$g(x)$	$f(g(x))$
-1	-1	**-4**	-4	**-7**
0	1	**2**	-1	**-1**
1	3	**8**	2	**5**
		*		*

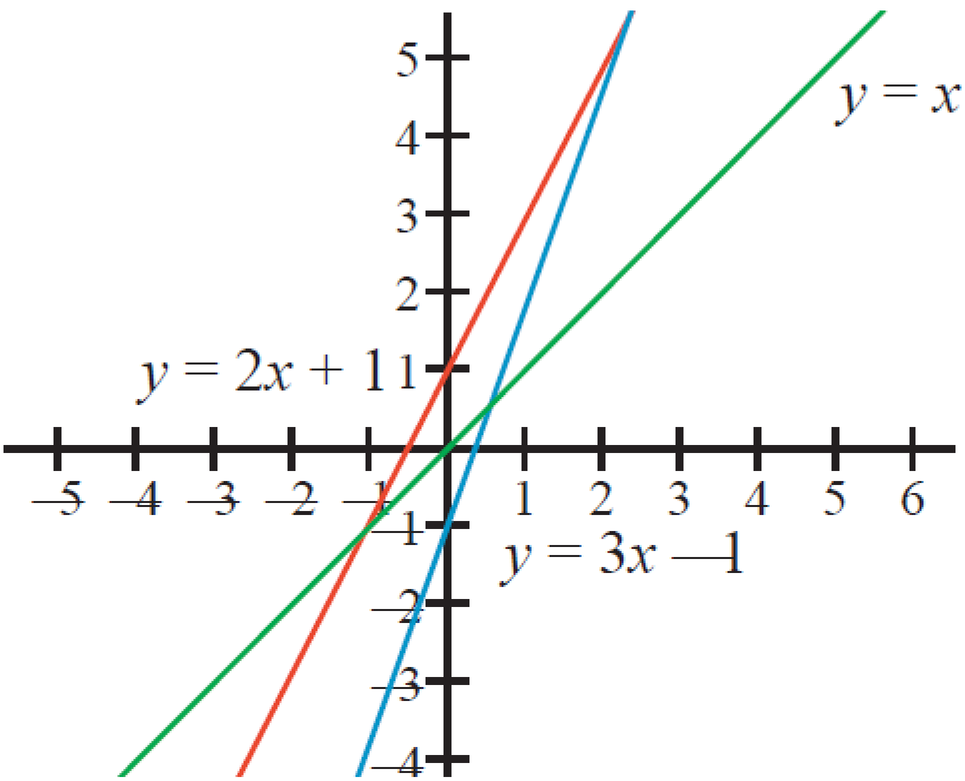
Compare
$g(f(x)) \neq f(g(x))$

$g(f(x)) = g(2x + 1) = 3(2x+1) - 1 = 6x + 3 - 1 = 6x + 2$

$f(g(x)) = f(3x - 1) = 2(3x-1) + 1 = 6x - 2 + 1 = 6x - 1$

$g(f(x)) \neq f(g(x))$

Visually note that the graphs of these two functions do not match up (mirror each other) when folded across the line y = x.



The composite functions shown above, f(g(x)) and g(f(x)), are not commutative.
$f(g(x)) \neq g(f(x))$.

Use function machines to teach about composite functions

Function machine for f(x)

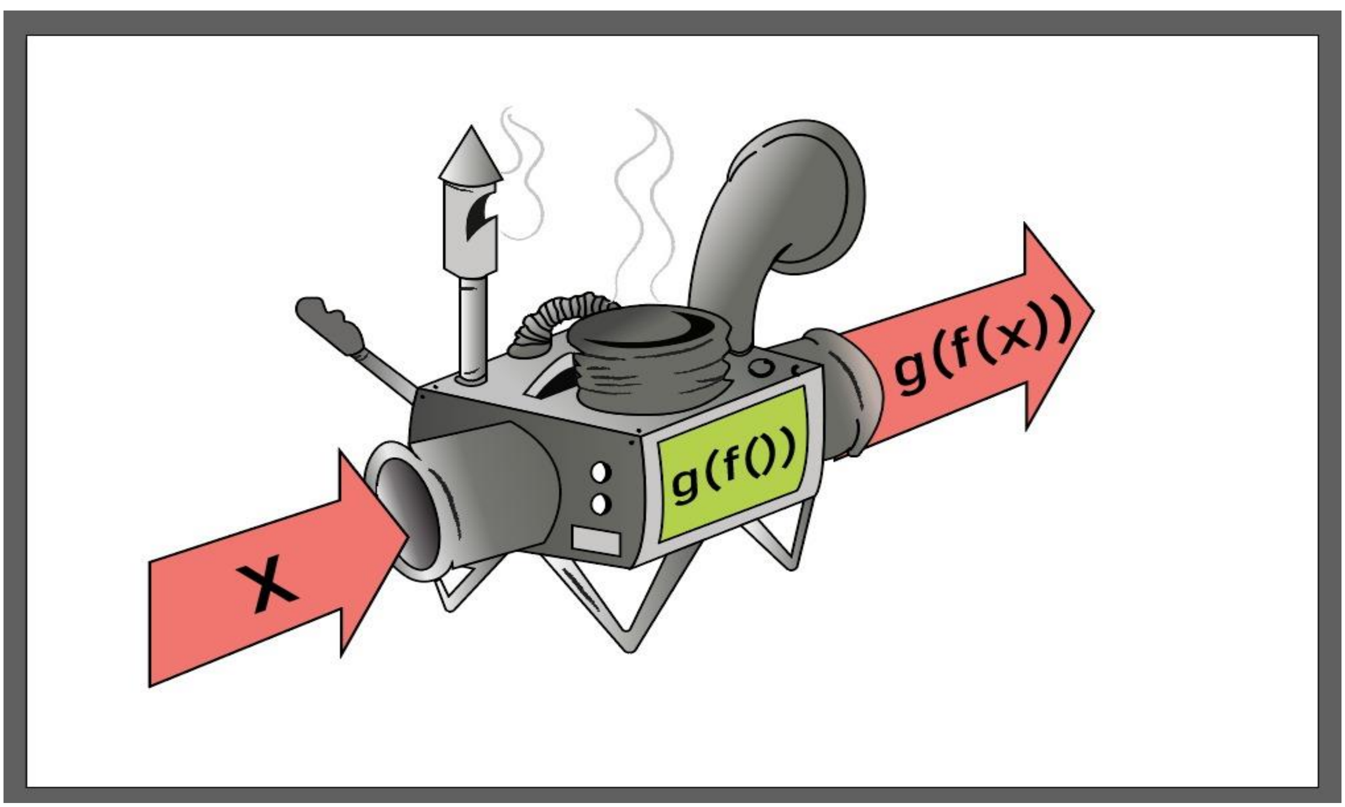
Function machine for g(x)

f(x) is input into g(x)

Or, composing function machines f(x) and g(x) as shown above, we get g(f(x)) below.

f(x) is calculated first and then it is then substituted into g(x).

Composition of Inverse Linear Functions



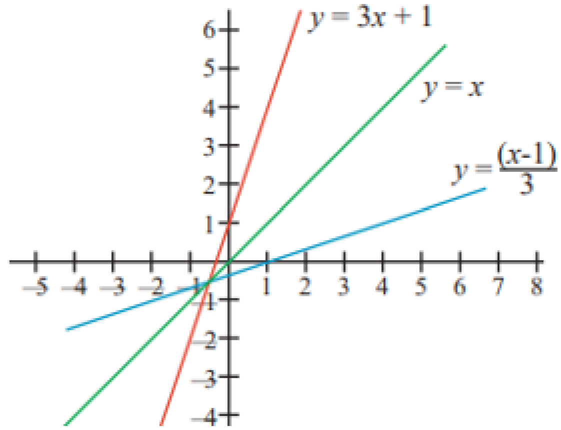
It was shown above for f(x) = 2x + 1 and g(x) = 3x – 1 that f(g(x)) ≠ g(f(x)). It is usually the case that composition of functions is not communitive. However, if f(x) and g(x) are inverse functions … let's see what happens. If y = 3x + 1 then its inverse would be x = 3y + 1… or y = $\frac{x-1}{3}$

Composition of Inverse Linear functions: y = 3x + 1 and x = 3y + 1

$f(x) = 3x + 1$ $\quad$ $g(x) = \frac{x-1}{3}$

x	$f(x)$	$g(f(x))$	$g(x)$	$f(g(x))$
−1	−2	**−1**	$-\frac{2}{3}$	**−1**
0	1	**0**	$-\frac{1}{3}$	**0**
1	4	**1**	0	**1**

Compare
$x = g(f(x)) = f(g(x))$



The two lines here match up when folded across the line y = x. When linear functions and their inverses are composed the result is the same… f(g(x)) = g(f(x)). Furthermore f(g(x)) = g(f(x)) = x. Composition of inverse functions results in the identity function y = x… f(x) = x.

f (g(x)) = f($\frac{x-1}{3}$) = 3($\frac{x-1}{3}$) + 1 = x

g(f(x)) = g(3x + 1) $\frac{(3x+1)-1}{3}$ = x

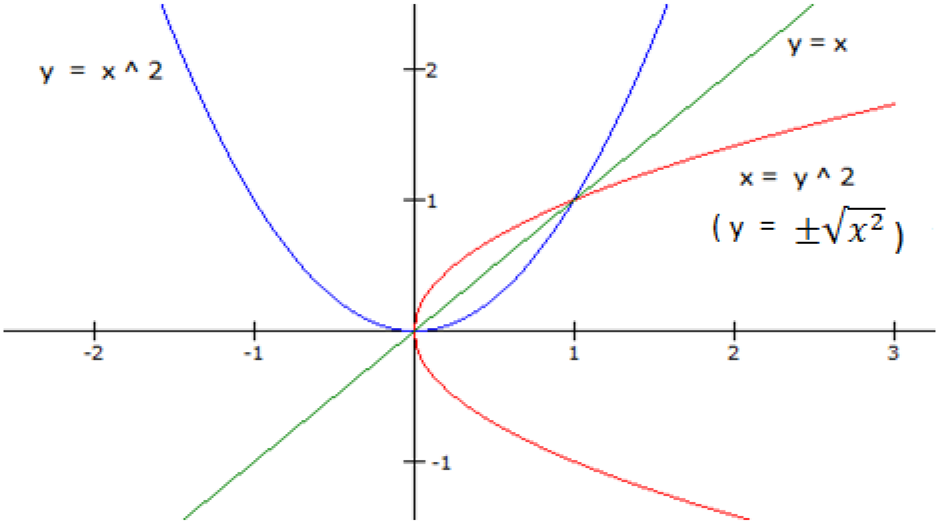
So f(g(x)) = g(f(x)) and they both equal x.

These ideas are repeated for special curvilinear functions.

Composition of Inverse Non-Linear Functions, #1

Compare the following two functions: y = x^2 and x = y^2 (or y = $\pm\sqrt{x}$)

Because the x and y of the two equations have been exchanged they are inverse relations. It follows that the two relations match up when reflected about the line $y = x$.



So let f(x) = x^2 and let g(x) = $\pm\sqrt{x}$

Then, because f(x) and g(x) are inverse relations they will match up when folded across the line y = x and

f(g(x)) = g(f(x)) = x	g(f(x)) = f(g(x)) = x
f($\pm\sqrt{x}$) = $(\pm\sqrt{x})^2$ $= \mathrm{x}$	g(x^2) = $\pm\sqrt{x^2}$ = $\pm$x

It should be noticed that f(x) and g(x) are inverse relations but not "inverse functions" because y = $\pm\sqrt{x^2}$ is not a function (does not pass the vertical line test). It should also be noticed that y = $\sqrt{x}$ only exists when x $\geq$ 0. Since composition of functions is only valid when both functions are defined then composition of the square and square root functions is only valid when x $\geq$ 0. Furthermore, since the square root of a negative number is imaginary then most applications composing y = x^2 and y = $\sqrt{x}$ will only happen when x $\geq$ 0 and y $\geq$ 0 (Quadrant 1).

Composition of Inverse Non-Linear Functions, #2

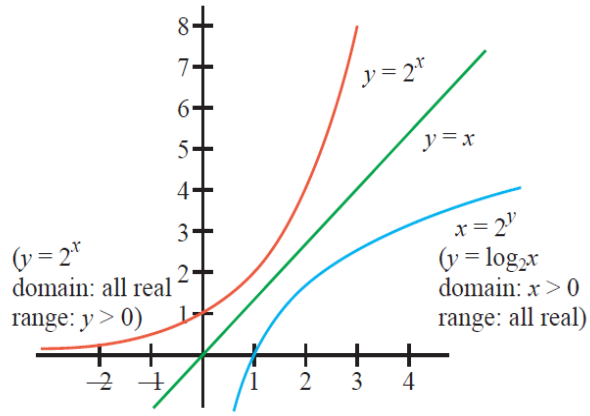
The graph at the right below shows the graphs of two functions—$y = 2^x$ and its inverse, $x = 2^y$—both plotted on the same x–y axis. Again notice that folding the graph along the line $y = x$ causes the two inverse functions to match up with each other. The two functions are symmetric around the line $y = x$. Notice the domain and range of $y = 2^x$ and notice that the domain and range restrictions have been exchanged for $x = 2^y$ (a.k.a. $y = \log_2 x$)

$y = 2^x$ (base $b > 1$)

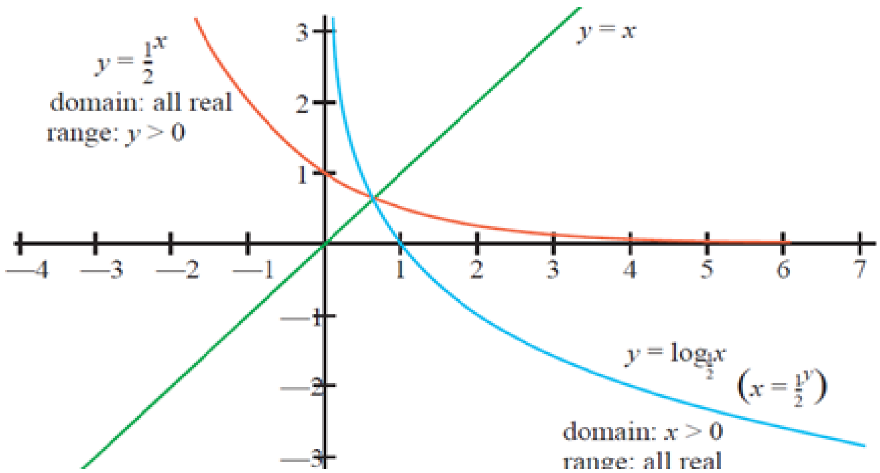
x	y
–2	¼
–1	½
0	1
1	2
2	4
3	8

$x = 2^y$ or $y = \log_2 x$ (base > 1)

x	Y
¼	–2
½	–1
1	0
2	1
4	2
8	3



Think of the graph b^y = x (a.k.a. y = $\log_b x$) as graphing b^x = y but with all the ordered pairs exchanged. These ideas are repeated for 0 < b < 1.



$y = \left(\frac{1}{2}\right)^x$

x	y
–2	4
–1	2
0	1
1	¼
2	½

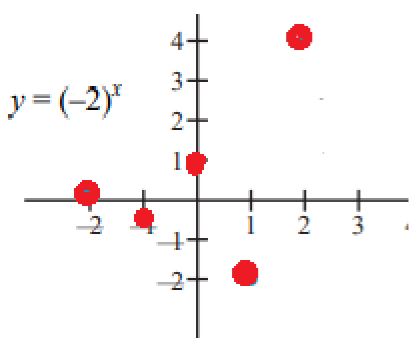
$x = \left(\frac{1}{2}\right)^y$ $\left(y = \log_{\frac{1}{2}} x\right)$

x	Y
4	–2
2	–1
1	0
¼	1
½	2

For the equation y = b^x the base b is not allowed to be negative. That would result in a discontinuous function. Also b ≠ 1 as y = 1^x would be boring and probably not applicable to anything useful.

$y = (-2)^x$

x	y
–2	¼
–1	–½
0	1
1	–2
2	4
3	–8

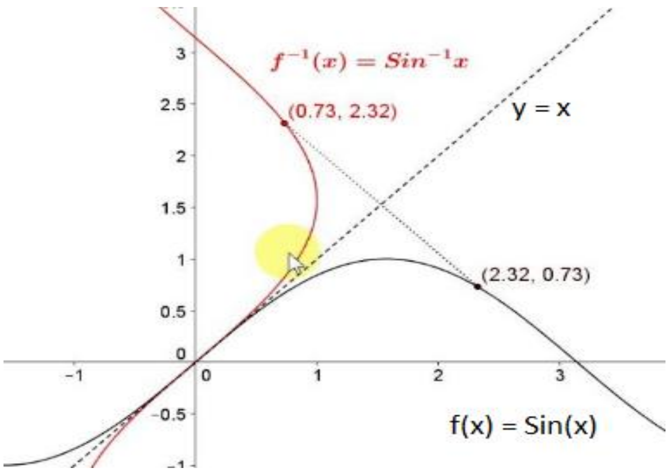


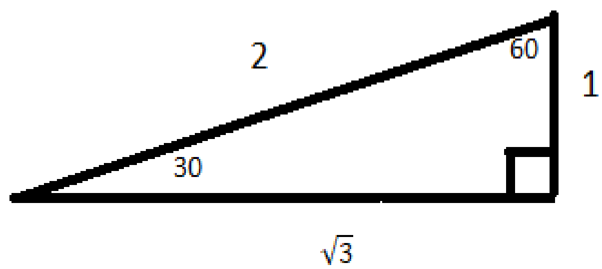
Let y = b^x ... x = b^y ... y = $\log_b x$ ...	f(x) = b^x g(x) = $\log_b x$	Let y = b^x ... x = b^y ... y = $\log_b x$ ...	f(x) = b^x g(x) = $\log_b x$
f(g(x)) = f($\log_b x$) = $b^{\log_b x}$ = x Because when f() and g() are inverse functions then f(g(x)) = x		g(f(x)) = g(b^x) = $\log_b b^x$ = x Because when f() and g() are inverse functions then g(f(x)) = x	

So $b^{log_b x}$ = **x and** $log_b b^x$ = **x** Composition of inverse functions results in the identity function, y = x… or f(x) = x.

Composition of Nonlinear Inverse Functions, Application #3

Let y = sin(x) then its inverse function would be x = sin(y)… i.e. y = sin^{-1}(x).





Since $sin^{-1}(x)$ only exists when -1 < x < 1 then sin(x) and sin^{-1} (x) can only be composed over that domain. Be careful if the angle is given in degree mode and your calculator is set to use radian measure. There will be compli-cations unless you first convert your angle measure into degree measure. Be sure to check on your calculator setting.

Let f(x) = sin^{-1}(x) and g(x) = sin(x)

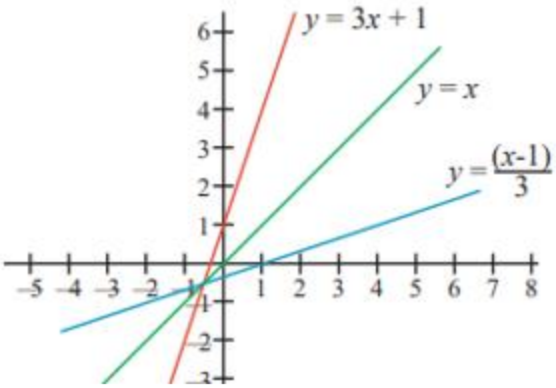
$sin^{-1}(\sin(30°))$ = $sin^{-1}(\frac{1}{2})$	=	30°
sin^{-1}(sin(X°))	=	x°

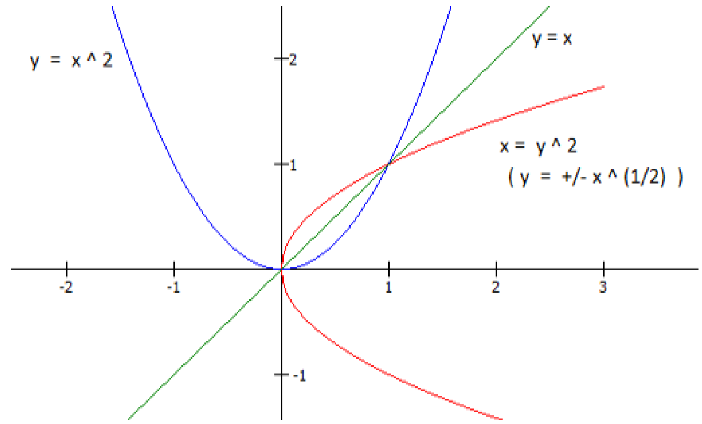
sin($sin^{-1}(\frac{1}{2})$)	=	sin(30°)	=	$\frac{1}{2}$
sin(sin^{-1}(x))	=			x

Note above if the argument for the sin() function is x° (not x radians) that sin^{-1}(sin(x)) ≠ sin($sin^{-1}(x)$). What is the difference? Why did that happen?

Composition of Nonlinear Inverse Functions, Application #4

Mathematics is much easier if you can apply the same idea over and over in different situations.





The area of a circle is 20 square inches.
What is its radius?

$A = \pi\, r^2$

$20 = 3.14159\, r^2$

$$\frac{20}{3.14159} = r^2$$

$$\sqrt{\frac{20}{3.14159}} = \sqrt{r^2}, \quad \sqrt{}\ and \quad {}^2$$

$$\sqrt{6.3662} = r$$

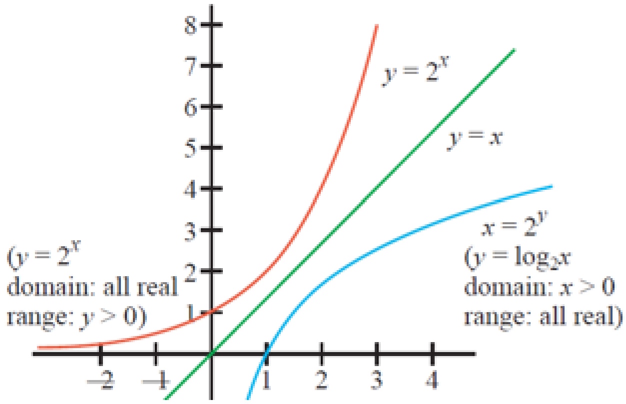
r = 2.5231

For inverse functions

$$f(g(x)) = g(f(x)) = x$$

Composition of Nonlinear Inverse Functions, Application #5

y = b^x inverse is x = b^y (i.e. y = $\log_2 x$)



For inverse functions

$$f(g(x)) = g(f(x)) = x$$

When g(x) = b^x and f(x) = $\log_b x$ then, because g(x) and f(x) are inverse functions, their composition results in the identity function.

$$log_b b^x = \mathbf{x} \quad \text{and} \quad b^{log_b x} = \mathbf{x}$$

Often the base of an exponential equation (y = b^x) is 10... i.e. y = 10^x. If that is the case the base of its inverse (x = b^y) need not be explicitly indicated as it is implied... $\log_{10} x$ = log x. Also if the base of an exponential equation y = b^x is "e"... y = e^x ... then the base of its inverse function (x = e^y) in logarithmic form need not be explicitly indicated as it is implied... $\log_e x$ = ln x. Therefore it follows that:

$$log_b b^x = \mathbf{x} \quad \text{and} \quad b^{log_b x} = \mathbf{x}$$

$$\log_e e^x = \boldsymbol{x} \quad \text{and} \quad e^{\log_e x} = x$$

$$\ln e^x = \mathbf{x} \quad \text{and} \quad e^{\ln x} = \mathbf{x}$$

Do you know what "e" is? It is a symbol that stands for an irrational number... an infinite, non-repeating number like π *or* φ. "e" is approximately 2.71828183...

Repeating from above "e" is approximately 2.71828183…

Advanced math teachers like to tell you that e = $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. That seems mysterious to many students. Let's demonstrate with a few values of n.

$$\left(1+\frac{1}{10}\right)^{10} = 2.593742460$$

$$\left(1+\frac{1}{100}\right)^{100} = 2.704813829$$

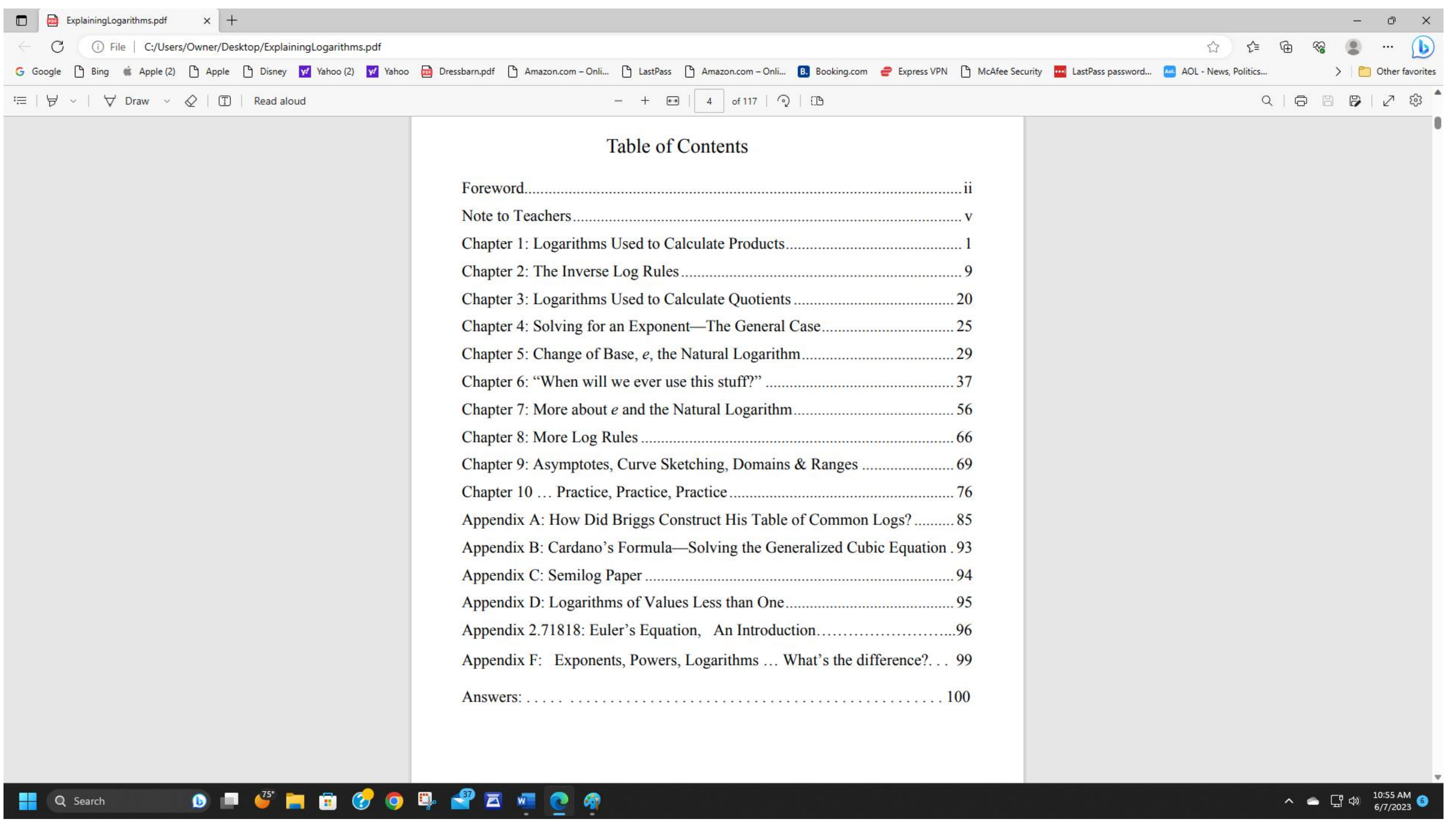
$$\left(1+\frac{1}{1,000}\right)^{1,000} = 2.716923932$$

$$\left(1+\frac{1}{10,000}\right)^{10,000} = 2.718145927$$

$$\left(1+\frac{1}{100,000}\right)^{100,000} = 2.718268237$$

$$\left(1+\frac{1}{1,000,000}\right)^{1,000,000} = 2.718280469$$

Hopefully this progression of values is more comfortable and less mysterious. While some students are comfortable with "being told" many students learn best by "being shown." Also it helps to see ideas being applied. Please refer to Chapters 5, 6 and 7 of Explaining Logarithms… www.mathlogarithms.com for applications of this mysterious "e."



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Composition of Nonlinear Inverse Functions, Application #6

This example of composition of inverse functions is used to solve a differential equation. If you are not familiar with differential equations then just skip on down to the part indicated by red stars below to see an application of $e^{\ln x} = \mathbf{x}$ shown immediately above.

Differential Equations in Meteorology—Barometric Pressure

It seems counterintuitive, but air has weight. A column of air above your head is pushing down on you right now. At sea level the weight of that column of air would be more than if you were standing on top of Mt. Everest because there would be a shorter column of air over your head on top of Mt. Everest than there would be at sea level. For points on the earth's surface, a simplistic model of barometric pressure, p (in inches of mercury in a barometer), is that, with increasing altitude, pressure decreases in direct proportion to the current pressure: $\frac{dp}{dh} = -0.2p$ where $p = 29.92$ inches of mercury at sea level where $h = 0$ (miles). Find the barometric pressure at the top of Mt. Everest at 29,029 ft.

$$\frac{dp}{dh} = h^0 \times -0.2p \qquad \text{form: } \frac{dp}{dh} = f(h) \times g(p)$$

$$\frac{dp}{p} = -0.2\,dh \qquad \text{separate variables}$$

$$p^{-1}\,dp = -0.2\,dh \qquad \frac{1}{p} = p^{-1}$$

$$\int p^{-1}\,dp = \int -0.2\,dh \qquad \text{integrate both sides}$$

$$\ln|p| + c_1 = -0.2h + c_2 \qquad \int p^{-1} = \ln|p|$$

$$\ln p = -0.2h + c_3 \qquad \text{combine constants of integration}$$

$$e^{\ln p} = e^{-0.2h+c_3} \qquad a = b \text{ so } e^a = e^b$$

$$p = e^{-0.2h} \times e^{c_3} \qquad e^{\ln p} = p$$

$$p = e^{-0.2h} \times c_4 \qquad \text{new constant ***}$$

$$29.92 = c_4 \times e^{-0.2\times 0} \qquad p = 29.92 \text{ at sea level}$$

$$29.92 = c_4 \times 1 \qquad e^0 = 1$$

$$c_4 = 29.92$$

$$p = 29.92 \times e^{-0.2h} \qquad \text{from *** above}$$

$$p = 29.92e^{-0.2\times\frac{29{,}029}{5{,}280}} \qquad \text{Everest's } h \text{ in miles}$$

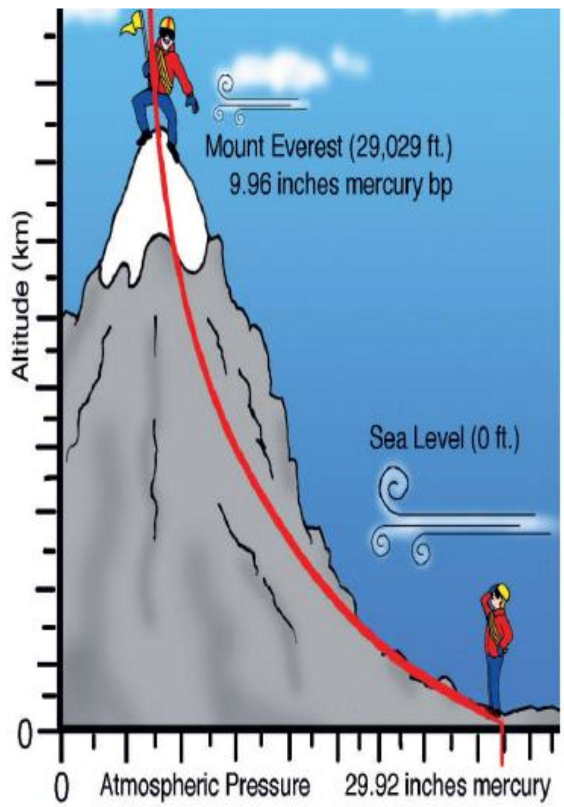
$$p = 29.92 \times e^{-0.2\times 5.497916667}$$

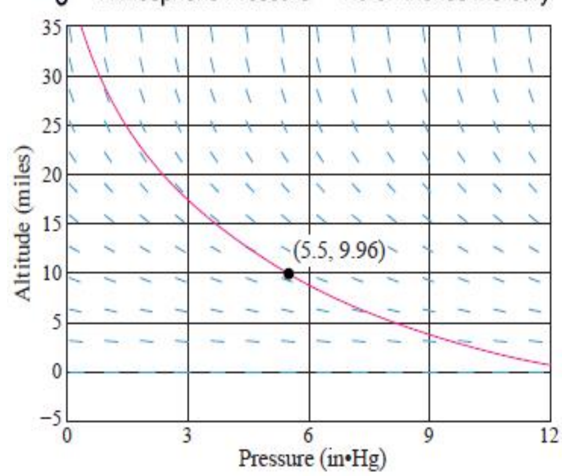
$$p = 29.92 \times e^{-1.099583333}$$

$$p = 29.92 \times 0.3330098089$$

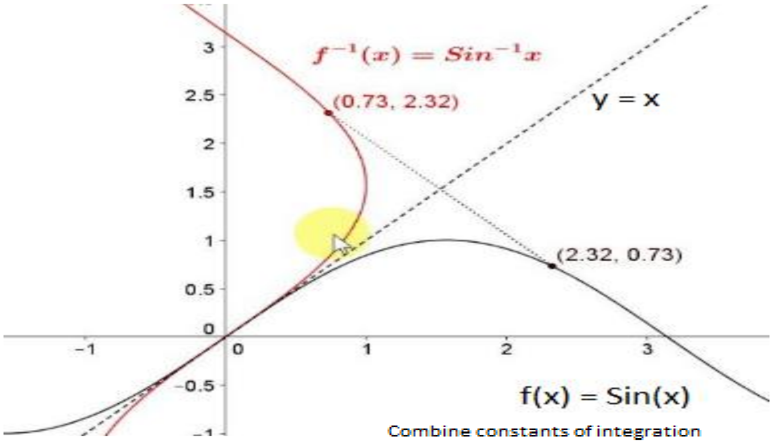
$$p = 9.96 \text{ inches of mercury}$$

Author's note: $\frac{29{,}029 \text{ ft}}{5{,}280 \text{ ft/mi}} = 5.5$ miles.

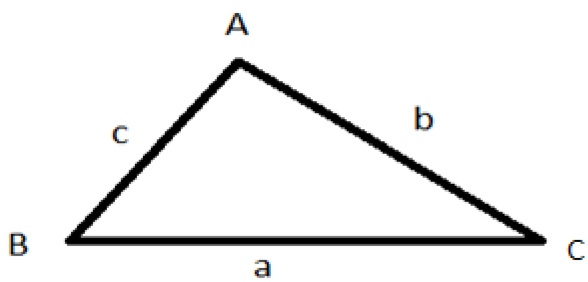


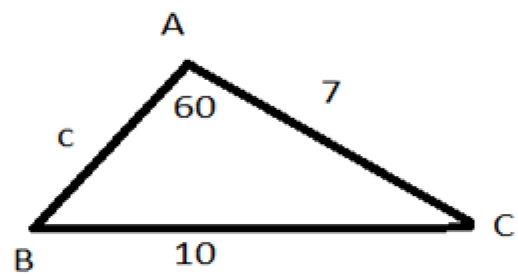


Composition of Nonlinear Functions, Application #7



Use the Law of Sines to determine angle B in the following figure.



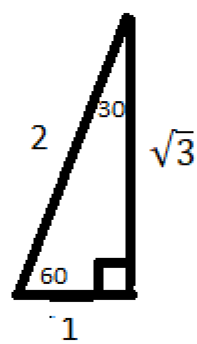


$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

$$\frac{10}{Sin\ 60°} = \frac{7}{Sin\ B}$$

10 Sin B = 7 Sin 60°

Sin B = $\frac{7}{10}$ Sin 60°



Be sure to set your calculator for degree mode

Sin B = $\frac{7}{10}\frac{\sqrt{3}}{2}$ (Sin 60° = $\frac{\sqrt{3}}{2}$)

Sin B = $\frac{12.12435}{20}$ = 0.60621778

Sin^{-1}(Sin B) = Sin^{-1}(0.60621778) **composition of inverse functions**

B = 37.3165236 °