Differential Equations: A Visual Introduction for Beginners

First printing

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Chapter 1
Getting Started

A differential equation is an equation in one or more variables involving one or more of its own derivatives. A major goal of taking a class in differential equations is to solve for \( f(x) \) if given \( f'(x) \), \( f''(x) \), etc. That is, somehow work backward and find the function whose derivative is given. For example, given \( f'(x) = 2x \), what is \( f(x) \)?

Once we obtain the mystery \( f(x) \), we can evaluate it for any desired \( x \). What would be the function whose derivative, \( \frac{dy}{dx} \), is \( 2x \)? We know from differential calculus that the derivative of \( y = x^2 \) would be \( 2x \). Hence, if \( \frac{dy}{dx} = 2x \) then \( y = x^2 \). (This can also be stated as: if \( f'(x) = 2x \), then \( f(x) = x^2 \).)

But wait! A vertical shift of a function does not impact the slope of tangent lines! In the figure at right, the tangent slopes to the curves \( y = x^2 - 1 \) (green), \( y = x^2 + 2 \) (red), and \( y = x^2 - 3 \) (blue) are all equal when \( x = a \), when \( x = 0 \), and when \( x = b \). There is not a single functional solution (one unique function whose derivative is \( 2x \)), but a set or family of them. We would indicate this by stating that for \( \frac{dy}{dx} = 2x \) the set or family of function solutions is \( x^2 + c \), where \( c \) represents any constant. The vertical lines here, \( x = a \), \( x = 0 \), and \( x = b \) are called isoclines. An isocline is notable because the slope of each function at each point on the isocline is the same. Hence, the goal of solving a differential equation is to solve for a function or family of functions which, when substituted into the original equation, will balance the two sides or make them equal.
From that family of function solutions we often, using information given us, will identify the particular one that is appropriate to our situation and use it to evaluate for specific values of $x$. That is called solving an initial value problem (IVP). In differential calculus, we studied how to obtain a derivative function from a given function. In differential equations, we study how to obtain a function from a given differential.

In a famous French play *Le Bourgeois Gentilhomme* by Molière, a comical but buffoonish character, Monsieur Jourdain, is amazed to learn that he had been speaking prose all his life and didn’t even know it.

“Par ma foi! Il y a plus de quarante ans que je dis de la prose sans que j’en susse rien, et je vous suis le plus obligé du monde de m’avoir appris cela.”

As Monsieur Jourdain demonstrates, it is quite possible to speak prose without knowing that you are doing it. However, it is very, very difficult to learn abstract skills and abstractions without, *at some conscious or unconscious level*, building upon earlier more primitive ones. Differential-equations teachers will not tell you the information that follows because they are so smart and they have internalized it so deeply that they assume that you have too. To be fair to them, the university curriculum requires that they move at a very fast pace, chop-chop. Nevertheless, it is a truism that mathematicians are not always “math educators.” Be grateful if you have a teacher who can move comfortably between the two worlds. Thank him or her.

It turns out that you have been practicing many of the necessary skills to work with and solve differential equation problems since algebra and you did not even know it. Making the connections shown in the following page or so of text will level the steep learning curve leading to differential equations and perhaps give you the confidence to say, “Wow! I have been doing much of this work since algebra and I did not even know it!”
There are many, many coplanar lines that can be drawn with the same linear slope. Lines that have the same slope are said to be parallel. They do not intersect.

There are many, many quadratic equations that can be drawn each with the same function slope. For example, \( \frac{dy}{dx} = 2x \). Coplanar curves with the same slope can be said to be “parallel.” They do not intersect.

Euclid’s famous fifth postulate suggests that only one of a set of parallel lines can pass through a given point.

The “Theorem of Uniqueness” says that only one of a family of “parallel” curves can pass through a given point.

We use the above assumption to identify one of a family of parallel lines. Find the equation of a line, \( y = mx + b \), that has a slope of 2 and passes through the point (2, 1).

We use the above theorem to identify one of a family of “parallel” parabolas. Find the equation of the parabola, \( y = x^2 + b \), that passes through the point (2, 1). Here the slope of \( y = x^2 + b \) is \( 2x \) because \( \frac{dy}{dx} (x^2 + b) = 2x \).

The line passing through (2, 1) with a slope of 2 is \( f(x) = 2x - 3 \).

The parabola passing through (2, 1) with a slope of 2 is \( f(x) = x^2 - 3 \).
It is possible to find the equation of the line with given slope and passing through a given point. You could then find the value of \( y \) for any given \( x \) on that line: \( y = f(x) \).

\[
y = 2x - 3 \rightarrow y = 2(6) - 3 \rightarrow y = 12 - 3 \rightarrow y = 9
\]

The line with slope 2 passing through (2, 1) also passes through (6, 9). Only one straight line does this.

It may be possible to find a specific function from a family of solutions to \( \frac{dy}{dx} \) that passes through a specific point. This is called solving an initial value problem (IVP). You could then find the \( y \) for any given \( x \) on that function, \( y = x^2 - 3 \).

\[
y = x^2 - 3 \rightarrow y = 6^2 - 3 \rightarrow y = 36 - 3 \rightarrow y = 33
\]

The parabola with slope \( 2x \) passing through (2, 1) also passes through (6, 33). Only one parabola does this. The complicated existence and uniqueness theorem addresses this in a real differential-equations class.

**Math is easier if you can make connections to previous topics! Differential equations can be seen as a curvilinear version of work done previously with linear functions.**

Or, as Monsieur Jourdain would have said, “Par ma foi! Il y a plus de quarante ans que je dis de la prose sans que j’en susse rien, et je vous suis le plus obligé du monde de m’avoir appris cela.”

In Chapter 1, the differential equation we have been using was \( \frac{dy}{dx} = 2x \) (or \( f'(x) = 2x \)). For that simple equation it was intuitive from beginning calculus that \( f(x) = x^2 \). Any differential equation of any consequence will not be solved by inspection. It is good to understand that working with differential equations is kind of a reverse process of differential calculus. Since the differential equation \( f'(x) \) or \( \frac{dy}{dx} = 2x \) was obtained by differentiating \( f(x) \), then perhaps you can anticipate that \( f(x) \) will be obtained by integrating \( f'(x) \).

For \( f(x) = x^2 - 3 \) the derivative \( f'(x) = 2x. \)

Therefore it follows that:

- \( \frac{dy}{dx} = 2x \)
- \( dy = 2x \, dx \)
- \( \int dy = \int 2x \, dx \)
- \( \int y^0 \, dy = 2 \int x^1 \, dx \)
- \( \frac{dy}{dx} = 2x \)
- \( y = \frac{x^2}{2} \)
- \( y = x^2 \)
- \( y = x^2 + c \)

There are many functions with \( \frac{dy}{dx} = 2x. \)

If it is given that the function \( y = x^2 + c \) passes through the point (2, 1), then

- \( 1 = 2^2 + c \)
- \( c = -3 \)
- \( y = x^2 + c \)
- \( y = x^2 - 3 \) equation of the parabola of form \( y = x^2 + b \)

and passing through (2, 1)

check: \( \frac{dy}{dx}(x^2 - 3) = 2x! \)

When \( x = 6 \), \( y = x^2 - 3 = 33. \)

Of all the functions whose derivative is \( \frac{dy}{dx} = 2x \), only one passes through the point (2, 1). That function also passes through the point (6, 33).
We are used to working with functions and function notation from both algebra and calculus. We have seen both symbols $f'(x)$ and $\frac{dy}{dx}$. However, up to now, that notation has mostly been used to evaluate a scalar or to indicate a function. For example, for $f(x) = x^2$, we have $f'(x) = 2x$ or $f'(5) = 10$. In differential equations, it will often be helpful to think of $f'(x)$, $\frac{dy}{dx}$, as an infinite set of tiny tangent segments, so tiny that each line segment is the length of a point. (Points don’t have length . . . use your imagination . . . think of a computer screen pixel.) Commercial computer software is available to create slope fields. Because the brain has a tendency to “fill in” gaps, you can, with imagination, “see” a finite representation of that desired set of functions that results from solving the “differential function.” The following is a progression of slope fields with $\Delta x = 1, 0.6, 0.3,$ and $0.1$. 

\[
\frac{dy}{dx} = 2x, \Delta x = 1
\]

\[
\frac{dy}{dx} = 2x, \Delta x = 0.6
\]

\[
\frac{dy}{dx} = 2x, \Delta x = 0.3
\]

\[
\frac{dy}{dx} = 2x, \Delta x = 0.1
\]
Putting all these ideas together into one marvelous MATLAB screen we see the slope field family prescribed
by the differential equation \( \frac{dy}{dx} = f'(x) = 2x \). We know that there is a specific function, \( f(x) \), somewhere in the
slope field family of functions whose derivative is \( 2x \), that passes through the point \((2, 1)\). Solving the differential
equation \( f'(x) \), we get \( f(x) = x^2 + c \) where \( c \) indicates an integration constant, a vertical shift of \( f(x) = x^2 \). Using
the information that the specific function we want passes through the point \((2, 1)\), we solved for \( c \) by substituting the
\((x, y)\) values \((2, 1)\) into \( y = x^2 + c \) to find \( c = -3 \). So \( y = x^2 + c \) becomes \( y = x^2 - 3 \). Then we could find, for \( x = 6 \),
\( y = 6^2 - 3 = 33!! \)

Do you remember in calculus how you learned to successively approximate the slope of a tangent
line by calculating the slope of approaching secant lines? (See Chapter 0.) Well a smart Swiss mathe-
matician, Leonhard Euler, 1707–1783, figured out a way to successively approximate the solution to
a differential equation for a given value of \( x \). The genius of Euler—who is credited with this iter-
tative method of solving for \( f(x) \) if given \( \frac{dy}{dx} \), that is \( f'(x) \)—was that 1) he could mentally visualize the
entire slope field for \( \frac{dy}{dx} \), that is every \( f(x) \) whose
derivative is \( f'(x) \), and 2) he realized that he could use the known fact that \((x, y)\) was on the unknown
\( f(x) \) to successively approach \( f(z) \) for any \( z \) using algebra and trigonometry and the known \( f'(x) \)—i.e., \( \frac{dy}{dx} \). Today, we have graphing calculators and computer software
packages that will help us see what Euler could visualize in the 1700s. As the slope indicators become shorter and
the tangent indicators become more numerous, each specific (particular) slope field approaches a particular solution,
\( f(x) \). Using the information we do have, \( \frac{dy}{dx} = f'(x) = 2x \) and the fact that \( f(x) \) passes through the point \((2, 1)\), we
can get an initial estimation of \( f(6) \) without knowing the function \( f(x) \).

In the figure above, we have a point \((2, 1)\) and we know the slope of the unknown function at \( x = 2 \) is 4. That is,
\( f'(2) = 2 \times 2 = 4 \). We wish to find the unknown point \((6, ?)\) on the unknown function \( f(x) \). With the two given \( x \)
values \((2 & 6)\), we can determine \( \Delta x = 6 - 2 = 4 \). If we could determine \( \Delta y \), we could calculate \( y + \Delta y \) and obtain
the new y value. How can we determine the Δy value? We know from studying trigonometry that \( \tan \theta = \frac{\text{opp}}{\text{adj}} \) hence \( \text{opp} = (\tan \theta) \times \text{adj} \). We know from studying algebra that \( m = \frac{\text{rise}}{\text{run}} = \frac{\text{opp}}{\text{adj}} \). So, \( \tan \theta = m \).

1. The slope \( m = \frac{y}{x} = \frac{\text{opp}}{\text{adj}} \).
2. \( \text{opp} = m \times \text{adj} \)

Substituting into \( \text{opp} = m \times \text{adj} \)

\[
\Delta y = m \times \Delta x
\]

\( \Delta y = 2x \times 4 \ldots m \) at point (2, 1) is 2x

\( \Delta y = 4 \times 4 = 16 \ldots m \) at point (2, 1) = 4

New \( y = \text{old } y + \Delta y \). \( y = 1 + 16 = 17 \). We estimate that \( f(6) \) for the unknown function, \( f(x) \), might be 17. Basically we have “extended” out (extrapolated) the tangent line until it reached an \( x \) value of 6.

Recall from differential calculus (reviewed in Chapter 0) how we improved our secant slope estimate of a tangent slope in the example by decreasing \( \Delta x \):

<table>
<thead>
<tr>
<th>( x ) (domain)</th>
<th>( y ) (range)</th>
<th>( m = \frac{y_2-y_1}{x_2-x_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( m = \frac{64-0}{8-0} = 8 )</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>( m = \frac{64-16}{8-4} = 12 )</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>( m = \frac{64-36}{8-6} = 14 )</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td>( m = \frac{64-49}{8-7} = 5 )</td>
</tr>
<tr>
<td>7.9</td>
<td>62.41</td>
<td>( m = \frac{64-62.41}{8-7.9} = 15.9 )</td>
</tr>
<tr>
<td>7.99</td>
<td>63.8401</td>
<td>( m = \frac{64-63.8401}{8-7.99} = 15.99 )</td>
</tr>
<tr>
<td>7.999</td>
<td>63.984001</td>
<td>( m = \frac{64-63.984001}{8-7.999} = 15.999 )</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
<td>( m = \frac{64-64}{8-8} = \frac{0}{0} = 16?? )</td>
</tr>
</tbody>
</table>

Indeterminate division by zero
That same “successive approximation” technique can be used in differential equations to improve our estimation of \( f(6) \). That is, given \( f'(x) = \frac{dy}{dx} = 2x \) and the fact that the function we seek passes through point \((2, 1)\), we can improve our original estimation of the unknown \( f(x) \) at \( 6 - f(6) \sim 17 \) by reducing the \( \Delta x \) we used to project out from \((2, 1)\).

For the example above, change from \( \Delta x = 4 \) to \( \Delta x = 1 \).

\[
\Delta x = 1, \quad \Delta y = 2 \times x_n \times \Delta x \quad \text{because} \quad \frac{dy}{dx} = 2x, \quad \text{so} \quad dy = 2x \times dx
\]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( x_n = x_{n-1} + \Delta x )</th>
<th>( \Delta y = 2 \times x_{n-1} \times \Delta x )</th>
<th>( y_n = y_{n-1} + \Delta y )</th>
<th>( (x_n, y_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 (given)</td>
<td>Not applicable</td>
<td>1 (given)</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>(3, 5)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>(4, 11)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
<td>11</td>
<td>(5, 19)</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10</td>
<td>19</td>
<td>(6, 29)</td>
</tr>
</tbody>
</table>

By decreasing \( \Delta x \), we get values closer and closer to the actual \( f(6) \). The text box at right and the five text boxes below were generated by the computer program shown on the following page.

Compare \((6, 32, 996)\) for \( \Delta x = 0.001 \), with the exact solution \((6, 33)\) we got several pages back. Fortunately, these answers are close. Otherwise, there would be egg on the author’s face!
There are software packages available that allow you to experiment with the ideas taught in a differential equations class. A jar file can be purchased from Cengage Publishers. One of the options allows the user to experiment with Euler’s method for different differential equations.

Below, passing through (0.24, 1.2), you see Euler’s method for $\Delta x = 1, 0.5, 0.25, 0.125$, and the Runge–Kutta 4 algorithm applied to the differential equation $\frac{dy}{dt} = y^2 - 4t$.

For people with a programming background it is not too much of a stretch to understand where the slope fields and the graphs of a particular function come from. Code based on Euler’s method (or perhaps another iterative algorithm called the Runge–Kutta method) is used on the given differential equation, $\frac{dy}{dx}$ or $f'(x)$, to generate a set...
of ordered pairs of points which, as closely as desired, approximate points lying on the unknown function, \( f(x) \). Then, using the slope information immediately available from the differential equation \( \frac{dy}{dx} = m \), right?) together with any desired delta value and simple trig function knowledge such as shown above in Euler’s method, one could write code to identify end points of a tangent segment to the unknown \( f(x) \). The code to draw straight-line segments between two points is well known and precoded in many languages (e.g., \texttt{g.drawLine} in Java). Voilà! There you have a tangent segment drawn to a point that is on, or very, very close to being on the unknown function curve. If you bring your delta close to zero, then the tangent-line slope segment approaches length “zero” which in “computerese” means one pixel. Then your graph would be a very close visualization of the unknown solution function itself. That function (shown above) can be thought of as a particular example of the family of solutions differentiated from the others by the fact that each of its tangent-slope segments are of length “one pixel.” Look at the figure and reread this paragraph if it was not clear at first.

**Chapter 1 Review**

Chapter 1 starts out defining a differential equation as an equation in one or more variables involving one or more of its own derivatives. A major goal of taking a class in differential equations is to solve for \( f(x) \) if given \( f'(x), f''(x), etc. \) That is, somehow work backward and find the function whose derivative is given. That paradigm is just the opposite of differential calculus when you are given a function and asked to find its derivative.

Side-by-side examples of coplanar parallel linear lines and coplanar “parallel” parabolas were given. It was shown that, if given a point on a straight line as well as its slope, you could find the \( y \) value of a different point on that line if given its \( x \) value. Similarly, it is the case for the parabolas. The concepts slope field and particular solution were introduced as well as Euler’s iterative technique for approximating an unknown point on an unknown function whose derivative and initial value are known.

**Online Application**

Visit demonstrations.wolfram.com/NumericalMethodsForDifferentialEquations for a great little application that will demonstrate some of the concepts from this chapter. “Numerical Methods for Differential Equations” from the Wolfram Demonstrations Project. Contributed by Edda Eich-Soellner.
Chapter 4
Solving Separable Differential Equations

In Chapter 3, we studied differential equations that, through algebraic techniques, could be rewritten in such a way that the numerator differential and its respective variable were together on one side of the equation and the denominator differential and its respective variable were together on the other side. Then, both sides were integrated.

Continuous Compounding of Interest
\[
\frac{dp}{dt} = r \times p
\]
\[
\frac{1}{p} dp = r\ dt
\]
\[
\int \frac{1}{p} dp = \int r\ dt
\]
and etc.

Continuous Population Growth
\[
\frac{dP}{dt} = r \times P
\]
\[
\frac{1}{P} dP = r\ dt
\]
\[
\int \frac{1}{P} dP = \int r\ dt
\]
and etc.

Carbon 14 Decay
\[
\frac{dC}{dt} = r \times C
\]
\[
\frac{1}{C} dC = r\ dt
\]
\[
\int \frac{1}{C} dC = \int r\ dt
\]
and etc.

Differential equations in which this technique can be used are called “separable differential equations.” They are the easiest type of differential equations to work with. Moving from general to specific,

\[
\frac{dy}{dx} = f(x) \times g(y), \quad g(y) \neq 0 \quad \text{standard form for a separable differential equation}
\]
\[
\frac{1}{g(y)}\ dy = f(x)\ dx
\]
\[
\int \frac{1}{g(y)}\ dy = \int f(x)\ dx
\]
and etc.,

we get the following examples.

<table>
<thead>
<tr>
<th>Ex 1</th>
<th>( \frac{dy}{dx} = 3 ) ( \frac{dy}{dx} = (3x^{0}) \times (1y^{0}) ) Separable!</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex 2</td>
<td>( \frac{dy}{dx} = \sqrt{x} \div \sqrt{y} ) ( \frac{dy}{dx} = \sqrt{x} \times \sqrt{y^{-1}} ) ( \frac{dy}{dx} = (x^{\frac{1}{2}}) \times (y^{-\frac{1}{2}}) ) Separable!</td>
</tr>
<tr>
<td>Ex 3</td>
<td>( \frac{dy}{dx} = x^{2} + 1 ) ( \frac{dy}{dx} = (x^{2} + 1) \times (1y^{0}) ) Separable!</td>
</tr>
<tr>
<td>Ex 4</td>
<td>( \frac{dy}{dx} = \sqrt{x} \div \sqrt{1-y^{2}} ) ( \frac{dy}{dx} = (x)^{2} \times (1-y^{2})^{-1} ) Separable!</td>
</tr>
<tr>
<td>Ex 5</td>
<td>( \frac{dy}{dx} = 3x^{2} + 4x + 2 \div 2(y-1) ) ( \frac{dy}{dx} = (3x^{2} + 4x + 2) \times \left[ \frac{1}{2(y-1)^{-1}} \right] ) Separable!</td>
</tr>
<tr>
<td>Ex 6</td>
<td>( \frac{dy}{dx} = \frac{y^{2} + 1}{x-1} ) ( \frac{dy}{dx} = (x-1)^{-1} \times (y^{2} + 1) ) Separable!</td>
</tr>
<tr>
<td>Ex 7</td>
<td>( \frac{dy}{dx} = \frac{y \cos x}{1 + 2y^{2}} ) ( \frac{dy}{dx} = (\cos x) \times \frac{y}{1 + 2y^{2}} ) Separable!</td>
</tr>
</tbody>
</table>
The point that is being made here is that if a differential equation can be put into the form
\[
\frac{dy}{dx} = f(x) \times g(y), \ g(y) \neq 0,
\]
then it can be solved by the “separable technique” as shown above. As a general rule, it is easier to immediately strive for separation of the variables and differentials from the beginning rather than to put the equation into the generic form shown here and then to separate. The example above is just to create an understanding of how to categorize differential equations.

Your differential-equations teacher sometimes has a perspective, an understanding and a schema that give an intangible advantage over you when looking at problems. Here is an example.

“Solving a separable differential equation is the reverse of implicit differentiation.”

So, what is a new skill to the student is just a backward version of what the differential equation teacher taught back in differential calculus. They blend seamlessly in the teacher’s mind.

If you understood that last sentence go ahead and skip over this next part. Otherwise let’s go back to differential calculus and review explicit and implicit differentiation and then revisit this idea. Find the slope of the tangent to the circle \(x^2 + y^2 = 25\) at the point when \(x = 4\). By substitution, then, \(4^2 + y^2 = 25\) and \(y = 3\). Using ideas taught in beginning calculus, then, the slope of the tangent would be equal to the derivative of the function evaluated at \((4, 3)\).
Explicit Differentiation
\[ x^2 + y^2 = 25 \text{ (25 is } r^2) \]
\[ y^2 = 25 - x^2 \]
\[ y = \pm \left(25 - x^2\right)^{\frac{1}{2}} \]
\[ \frac{dy}{dx} = \pm \frac{1}{2} \left(25 - x^2\right)^{-\frac{1}{2}} (-2x) \]
\[ \frac{dy}{dx} = \pm \frac{x}{\sqrt{25 - x^2}} \]
\[ \frac{dy}{dx} = \frac{-x}{y} \text{ (from *** above)} \]

Implicit Differentiation
\[ x^2 + y^2 = 25 \]
\[ \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \]
\[ 2x + 2y \frac{dy}{dx} = 0 \]
\[ 2y \frac{dy}{dx} = -2x \]
\[ \frac{dy}{dx} = \frac{-x}{y} \]
Here, the derivative is in terms of both \( x \) and \( y \).

Reverse Implicit Differentiation
\[ \frac{dy}{dx} = \frac{-x}{y} \]
\[ y \frac{dy}{dx} = -x \frac{dx}{dx} \]
\[ \int y \frac{dy}{dx} = - \int x \frac{dx}{dx} \]
\[ \frac{y^2}{2} + c_1 = - \frac{x^2}{2} + c_2 \]
\[ \frac{x^2}{2} + \frac{y^2}{2} = C \]
\[ x^2 + y^2 = K^2 \]
For \( x = 4 \) and \( y = 3 \), \( K = 5 \).

“Solving a separable differential equation is the reverse of implicit differentiation.”

Understanding connections makes mathematics more interesting. It also lets you see that new ideas are often connected to or related to old ones.

We’ll spend the rest of this chapter reviewing those slope fields. Since this is otherwise a short chapter and slope fields will be used extensively in the next chapter, it might be good to talk more about them again. Slope fields are useful in anticipating solutions to differential equations and are a necessity when trying to get information about unsolvable differential equations. Today’s software graphing packages really do all the work for you. However, like many topics in math, they obscure what really goes on. Just like everything else in postcalculator mathematics education, those graphing packages have potential for harm. For example, in my first book, Explaining Logarithms, I posited the scenario of a student claiming that \( 234 \times 4,192 = 8,219 \) or \( y = \log_{4.8} 714.6, \ y = 22.9 \) because “the calculator said so.”
All high school math teachers have heard statements such as these. As another example, when I was in college, the education majors would take ½-inch thick SAS or SPSS printouts off the printer and proceed to make incredible conclusions based on their “data.” When asked what a correlation or standard deviation was, they didn’t know and they did not care that they didn’t know.

It is best for most people to see how slope fields arise by doing a few of them the old fashioned way. We’ll start with a quick review from trig regarding selected slope values before proceeding. The following values are obtained by taking the tangents of 180, 157.5, 135, 112.5, 90, 67.5, 45, 22.5, and 0 degrees.

The following shows three different examples of slope fields being graphed “by hand” (without a computer).

1. Graph the slope field for $\frac{dy}{dx} = -\frac{x}{y}$

Step 1: Red slope segments for $x = 0$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>indet</td>
<td>red</td>
</tr>
<tr>
<td>(0, -1)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, -2)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, -3)</td>
<td>0</td>
<td>red</td>
</tr>
</tbody>
</table>

Step 2: Green slope segments for $x = 1$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 3)</td>
<td>$-\frac{1}{3}$</td>
<td>green</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>$-\frac{1}{2}$</td>
<td>green</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>-1</td>
<td>green</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>undef</td>
<td>green</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>1</td>
<td>green</td>
</tr>
<tr>
<td>(1, -2)</td>
<td>$\frac{1}{2}$</td>
<td>green</td>
</tr>
<tr>
<td>(1, -3)</td>
<td>$\frac{1}{3}$</td>
<td>green</td>
</tr>
</tbody>
</table>

Step 3: Blue slope segments for $x = -1$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1, 3)</td>
<td>$\frac{1}{3}$</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 2)</td>
<td>$\frac{1}{2}$</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 1)</td>
<td>1</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 0)</td>
<td>undef</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -1)</td>
<td>-1</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -2)</td>
<td>$-\frac{1}{2}$</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -3)</td>
<td>$-\frac{1}{3}$</td>
<td>blue</td>
</tr>
</tbody>
</table>
Step 4: Brown slope segments for $x = 2$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$ (m)</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 3)$</td>
<td>$-\frac{2}{3}$</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$-1$</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$-2$</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, 0)$</td>
<td>undefined</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, -1)$</td>
<td>$2$</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, -2)$</td>
<td>$1$</td>
<td>brown</td>
</tr>
<tr>
<td>$(2, -3)$</td>
<td>$\frac{2}{3}$</td>
<td>brown</td>
</tr>
</tbody>
</table>

Step 5: Orange slope segments for $x = -2$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$ (m)</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-2, 3)$</td>
<td>$\frac{2}{3}$</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, 2)$</td>
<td>$1$</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, 1)$</td>
<td>$2$</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, 0)$</td>
<td>undefined</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, -1)$</td>
<td>$-2$</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, -2)$</td>
<td>$-1$</td>
<td>orange</td>
</tr>
<tr>
<td>$(-2, -3)$</td>
<td>$-\frac{2}{3}$</td>
<td>orange</td>
</tr>
</tbody>
</table>

Step 6: Purple slope segments for $x = 3$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$ (m)</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3, 3)$</td>
<td>$-1$</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, 2)$</td>
<td>$-\frac{3}{2}$</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td>$-3$</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, 0)$</td>
<td>undefined</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, -1)$</td>
<td>$3$</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, -2)$</td>
<td>$\frac{3}{2}$</td>
<td>purple</td>
</tr>
<tr>
<td>$(3, -3)$</td>
<td>$1$</td>
<td>purple</td>
</tr>
</tbody>
</table>

Step 7: Black slope segments for $x = -3$.

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\frac{dy}{dx} = -\frac{x}{y}$ (m)</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-3, 3)$</td>
<td>$1$</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, 2)$</td>
<td>$\frac{3}{2}$</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, 1)$</td>
<td>$3$</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, 0)$</td>
<td>undefined</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, -1)$</td>
<td>$-3$</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, -2)$</td>
<td>$-\frac{3}{2}$</td>
<td>black</td>
</tr>
<tr>
<td>$(-3, -3)$</td>
<td>$-1$</td>
<td>black</td>
</tr>
</tbody>
</table>

What you need to do is to try to imagine many, many slope indicators each with diminished length until they reach the length of a point (points actually do not have length, think pixel). If you are successful in your imagination, you should be able to “see” (in your mind) the graph of the original function $f(x)$ that, when differentiated, resulted in $\frac{dy}{dx}$: concentric circles.
2. Graph the slope field for \( \frac{dy}{dx} = y - x \)

**Step 1: Red slope segments for \( x = 0 \).**

<table>
<thead>
<tr>
<th>( (x, y) )</th>
<th>( \frac{dy}{dx} = y - x ) ( (m) )</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3)</td>
<td>3</td>
<td>red</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>2</td>
<td>red</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>1</td>
<td>red</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>(0, -1)</td>
<td>-1</td>
<td>red</td>
</tr>
<tr>
<td>(0, -2)</td>
<td>-2</td>
<td>red</td>
</tr>
<tr>
<td>(0, -3)</td>
<td>-3</td>
<td>red</td>
</tr>
</tbody>
</table>

![Slope field for \( x = 0 \)]

**Step 2: Green slope segments for \( x = 1 \).**

<table>
<thead>
<tr>
<th>( (x, y) )</th>
<th>( \frac{dy}{dx} = y - x ) ( (m) )</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 3)</td>
<td>2</td>
<td>green</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>1</td>
<td>green</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>0</td>
<td>green</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>-1</td>
<td>green</td>
</tr>
<tr>
<td>(1, -1)</td>
<td>-2</td>
<td>green</td>
</tr>
<tr>
<td>(1, -2)</td>
<td>-3</td>
<td>green</td>
</tr>
<tr>
<td>(1, -3)</td>
<td>-4</td>
<td>green</td>
</tr>
</tbody>
</table>

![Slope field for \( x = 1 \)]

**Step 3: Blue slope segments for \( x = -1 \).**

<table>
<thead>
<tr>
<th>( (x, y) )</th>
<th>( \frac{dy}{dx} = y - x ) ( (m) )</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1, 3)</td>
<td>4</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 2)</td>
<td>3</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 1)</td>
<td>2</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, 0)</td>
<td>1</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -1)</td>
<td>0</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -2)</td>
<td>-1</td>
<td>blue</td>
</tr>
<tr>
<td>(-1, -3)</td>
<td>-2</td>
<td>blue</td>
</tr>
</tbody>
</table>

![Slope field for \( x = -1 \)]
Step 4: Brown slope segments for \( x = 2 \).

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\frac{dy}{dx} = y - x ) ((m))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 3)</td>
<td>1</td>
<td>brown</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>0</td>
<td>brown</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>-1</td>
<td>brown</td>
</tr>
<tr>
<td>(2, 0)</td>
<td>-2</td>
<td>brown</td>
</tr>
<tr>
<td>(2, -1)</td>
<td>-3</td>
<td>brown</td>
</tr>
<tr>
<td>(2, -2)</td>
<td>-4</td>
<td>brown</td>
</tr>
<tr>
<td>(2, -3)</td>
<td>-5</td>
<td>brown</td>
</tr>
</tbody>
</table>

Step 5: Orange slope segments for \( x = -2 \).

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\frac{dy}{dx} = y - x ) ((m))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2, 3)</td>
<td>5</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, 2)</td>
<td>4</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, 1)</td>
<td>3</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, 0)</td>
<td>2</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, -1)</td>
<td>1</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, -2)</td>
<td>0</td>
<td>orange</td>
</tr>
<tr>
<td>(-2, -3)</td>
<td>-1</td>
<td>orange</td>
</tr>
</tbody>
</table>

Step 6: Purple slope segments for \( x = 3 \).

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\frac{dy}{dx} = y - x ) ((m))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 3)</td>
<td>0</td>
<td>purple</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>-1</td>
<td>purple</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>-2</td>
<td>purple</td>
</tr>
<tr>
<td>(3, 0)</td>
<td>-3</td>
<td>purple</td>
</tr>
<tr>
<td>(3, -1)</td>
<td>-4</td>
<td>purple</td>
</tr>
<tr>
<td>(3, -2)</td>
<td>-5</td>
<td>purple</td>
</tr>
<tr>
<td>(3, -3)</td>
<td>-6</td>
<td>purple</td>
</tr>
</tbody>
</table>

Step 7: Black slope segments for \( x = -3 \).

<table>
<thead>
<tr>
<th>((x,y))</th>
<th>(\frac{dy}{dx} = y - x ) ((m))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3, 3)</td>
<td>6</td>
<td>black</td>
</tr>
<tr>
<td>(-3, 2)</td>
<td>5</td>
<td>black</td>
</tr>
<tr>
<td>(-3, 1)</td>
<td>4</td>
<td>black</td>
</tr>
<tr>
<td>(-3, 0)</td>
<td>3</td>
<td>black</td>
</tr>
<tr>
<td>(-3, -1)</td>
<td>2</td>
<td>black</td>
</tr>
<tr>
<td>(-3, -2)</td>
<td>1</td>
<td>black</td>
</tr>
<tr>
<td>(-3, -3)</td>
<td>0</td>
<td>black</td>
</tr>
</tbody>
</table>
The slope field at right shows sample slopes of the differential equation \( \frac{dy}{dx} = y - x \). As indicated in the graph, there are many functions that have a slope of \( y - x \). If you are given that the graph of \( f(x) \) that you want passes through a specific point, say \((-3, -2)\), that will help you to identify a specific function, \( f(x) \), from the family of functions. This is shown in the figure at right. The progression of ideas is as follows: You are given a differential equation, \( f'(x) \), or the problem you are working on dictates a differential equation. The solution to that differential equation would be an entire family of possibilities, \( f(x) + c \), where \( c \) is unknown. You may be given (or the problem that you are working on will dictate) that the solution needed for your case will pass through a specified point. You are then to use the fact that the general solution, \( f(x) + c \), of the given differential equation, \( f' \), passes through the given point in order to solve for \( c \). From that \( f(x) + c \) with known \( c \), you could then find the value of \( f(x) \) when \( x = 2 \) or perhaps you would be asked to find the \( x \) or \( y \)-intercept of the particular \( f(x) \) passing through \((-3, -2)\) that is indicated by the differential equation \( f'(x) \), also known as \( \frac{dy}{dx} \). This is called an initial value problem (IVP) and is possible due to something called the “existence and uniqueness theorem.”

3. Graph the slope field for \( \frac{dy}{dx} = y^3 - y^2 - 6y \)

This example is shown because it demonstrates a differential equation with no explicit \( x \) variables. There is a fancy word for such equations: autonomous. A differential equation with no explicit \( x \) values is called autonomous. This fact will have an impact on the hand-graphing process: If there is no explicit \( x \) value in the differential equation, all the \( y \) values will be the same for any given \( x \). The following autonomous differential equation was chosen because the polynomial in \( y \) will factor, making for a nice clean example. \( y^3 - y^2 - 6y = y(y^2 - y - 6) = y(y - 3)(y + 2) \). Regardless of the \( x \) value, the derivative (slope) will be zero when \( y = -2, 0, \) and 3.

Start with the zero-slope parts of the field.

<table>
<thead>
<tr>
<th>All ( x ) values</th>
<th>( y )</th>
<th>( \frac{dy}{dx} = y^3 - y^2 - 6y = y(y - 3)(y + 2) )</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>\textcolor{red}{red}</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>\textcolor{red}{red}</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>\textcolor{red}{red}</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Now calculate several \(y\)-value slopes at once.

<table>
<thead>
<tr>
<th>All (x) values</th>
<th>(y)</th>
<th>(\frac{dy}{dx} = y^3 - y^2 - 6y = y(y - 3)(y + 2))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(4)</td>
<td>(4)(4 - 3)(4 + 2) = +24</td>
<td>green</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>2</td>
<td>(2)</td>
<td>(2)(2 - 3)(2 + 2) = -8</td>
<td>blue</td>
</tr>
<tr>
<td>1</td>
<td>(1)</td>
<td>(1)(1 - 3)(1 + 2) = -6</td>
<td>blue</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>-1</td>
<td>(-1)</td>
<td>(-1)(-1 - 3)(-1 + 2) = +4</td>
<td>green</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>-3</td>
<td>(-3)</td>
<td>(-3)(-3 - 3)(-3 + 2) = -18</td>
<td>blue</td>
</tr>
</tbody>
</table>

Finally, since slope lines are “parallel” (do not intersect each other), we can anticipate that the slope lines will approach the horizontal slope lines at \(y = -2, 0,\) and 3 asymptotically. Actually, since the curved slope lines cannot intersect each other either, each of those (curved) lines will approach each other asymptotically as well.

<table>
<thead>
<tr>
<th>All (x) values</th>
<th>(y)</th>
<th>(\frac{dy}{dx} = y^3 - y^2 - 6y = y(y - 3)(y + 2))</th>
<th>Slope color</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(4)</td>
<td>(4)(4 - 3)(4 + 2) = +24</td>
<td>green</td>
</tr>
<tr>
<td>3.1</td>
<td>(3.1)</td>
<td>(3.1)(3.1 - 3)(3.1 + 2) = (3.1)(0.1)(5.1) = 1.6</td>
<td>green</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>2.9</td>
<td>(2.9)</td>
<td>(2.9)(2.9 - 3)(2.9 + 2) = (2.9)(-0.1)(4.9) = -1.4</td>
<td>blue</td>
</tr>
<tr>
<td>2</td>
<td>(2)</td>
<td>(2)(2 - 3)(2 + 2) = -8</td>
<td>blue</td>
</tr>
<tr>
<td>1</td>
<td>(1)</td>
<td>(1)(1 - 3)(1 + 2) = -6</td>
<td>blue</td>
</tr>
<tr>
<td>0.1</td>
<td>(0.1)</td>
<td>(0.1)(0.1 - 3)(0.1 + 2) = (0.1)(-2.9)(2.1) = -0.6</td>
<td>blue</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>-0.1</td>
<td>(-0.1)</td>
<td>(-0.1)(-0.1 - 3)(-0.1 + 2) = (-0.1)(-3.1)(1.9) = 0.6</td>
<td>green</td>
</tr>
<tr>
<td>-1</td>
<td>(-1)</td>
<td>(-1)(-1 - 3)(-1 + 2) = +4</td>
<td>green</td>
</tr>
<tr>
<td>-1.9</td>
<td>(-1.9)</td>
<td>(-1.9)(-1.9 - 3)(-1.9 + 2) = (-1.9)(-4.9)(0.1) = 0.9</td>
<td>green</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>red</td>
</tr>
<tr>
<td>-2.1</td>
<td>(-2.1)</td>
<td>(-2.1)(-2.1 - 3)(-2.1 + 2) = (-2.1)(-5.1)(-0.1) = -1.1</td>
<td>blue</td>
</tr>
<tr>
<td>-3</td>
<td>(-3)</td>
<td>(-3)(-3 - 3)(-3 + 2) = -18</td>
<td>blue</td>
</tr>
</tbody>
</table>

Except for instructional materials on beginning differential equations, there is very little need to graph a lot of slope fields by hand. In this third case, graphing the slope field for \(\frac{dy}{dx} = y^3 - y^2 - 6y\), there is the benefit in that we get a visual on what the term autonomous means. Above, the term autonomous differential equation was defined as a differential equation that did not have any \(x\) terms. We can now add to that definition. An autonomous equation is a differential equation that, for each \(y\), has the same slope, \(\frac{dy}{dx}\), for all \(x\) values. This is also a good time to talk about sources and sinks. A source asymptote \((y = 3, y = -2\) at right) is a line from which the slope lines on each side are “leaving.” A sink asymptote \((y = 0)\) is a line that is being approached by slope lines on both sides. Note that approaching and leaving are taken from the point of view of \(x\) increasing.
A node asymptote has slopes approaching from one side but leaving from the other as shown in this unrelated graph. Another benefit to hand graphing an autonomous differential equation (e.g., \( \frac{dy}{dx} = y^3 - y^2 - 6y \)) is to introduce the concept of phase-plane analysis. Above, \( \frac{dy}{dx} = y^3 - y^2 - 6y \) was factored as \( \frac{dy}{dx} = y(y - 3)(y + 2) \). That looks very much like the polynomials we studied in precalculus. We remember graphing cubic polynomials. Often they crossed the independent axis at three places. This cubic polynomial would obviously cross at \(-2, 0, \) and \(3\). That suggests the following.

Graph \( y' = y^3 - y^2 - 6y \) just as you were taught in precalculus except, since all the independent and dependent values have changed, you would label the vertical and horizontal axes accordingly: the vertical axis as \( y' \) and the horizontal axis as \( y \) (figure below left). Since the coefficient of the \( y^3 \) term is positive, the graph will increase from negative infinity, cross at \( y = -2 \), then ascend above the \( y \)-axis until it reaches a local maximum where it descends and crosses again at \( y = 0 \). It then continues descending below the \( y \)-axis until it reaches a local minimum where it again starts ascending until it crosses the \( y \)-axis at \( y = 3 \) and continues on to positive infinity. For each place where the graph of \( y' = y^3 - y^2 - 6y \) crosses the \( y \)-axis, the graph of the solution to our differential equation, \( \frac{dy}{dx} = y^3 - y^2 - 6y \), would have a slope value of zero at that \( y \) value. This is shown in the \( x-y \) graph at the right.

Well alrighty now! The \( y-y' \) graph in the phase-plane analysis indicates that the slopes are all negative and asymptotic to \( y = -2 \) in the interval \((-\infty, -2)\) and all positive and asymptotic to \( y = 3 \) in the interval \((3, \infty)\). All the graphs in both those intervals get steep very fast as they approach infinity. (See the middle figure below.) In the \( y-y' \) graph in the phase-plane analysis, slopes in the interval \((0, 3)\) are all negative and cannot pass the lines \( y = 3 \) and \( y = 0 \) and hence are asymptotic to them both. Similarly, this can be said for the slopes in the interval \((-2, 0)\) except that the slopes are positive in that interval. (See figure below right and compare it with the one at the bottom of page 51.)
Chapter 4 Review

Chapter 4 showed how to solve a type or category of differential equation called separable differential equations. Here, the numerator and denominator of the derivative of the differential equation were separated by multiplying both sides of the differential equation by the denominator of the derivative. Additionally, the dependent and independent variables were moved to the appropriate sides of the equation using algebra. Then, the respective independent and dependent variables are obtained by integrating their respective differentials.

We learned to identify separable differential equations from their generic form, which either looked like or could be transformed to look like \( \frac{dy}{dx} = f(x) \times g(y) \), \( g(y) \neq 0 \).

It was shown that solving a separable differential equation was the reverse of implicit differentiation. As a learning exercise, “slope fields” for three different differential equations were drawn in stages “by hand.” In practice, these slope fields just magically appear in Mathematica or MATLAB, but it is helpful in the learning process to draw a few slope fields the old fashioned way to take the magic out of the process.

Several new vocabulary words were introduced:

1. Autonomous differential equations—a differential equation that does not have any \( x \) terms
2. Source asymptote—an asymptote from which a particular solution to a differential equation is diverging
3. Sink asymptote—an asymptote toward which a particular solution is converging
4. Node asymptote—an asymptote that has a particular solution converging on one side but diverging from the other
5. Phase-plane analysis—a technique for studying the behavior of a differential equation. With a differential equation the axis is labeled using the dependent and independent variables of the equation. With the phase-plane analysis the axis of the graph is labeled using the dependent variable (of the differential equation) as the horizontal axis and the derivative of the dependent variable (of the differential equation) as the vertical axis.

Online Application

Chapter 5
More Applied Separable Differential Equations

Differential Equations in Chemistry

In a chemical reaction, 100 g of substance A are being dissolved at a rate that is directly proportional to the square of the undissolved amount remaining. After one hour, only 20 g of substance A remain undissolved. How much of substance A is present after two hours? How long before the amount of substance A decreases to 1 g?

Let \( A \) be the amount of substance A at any time \( t \). From the data and assumption about the rate of dissolution, you can write the following differential equation:

\[
\frac{dA}{dt} = kA^2.
\]

Convert this differential equation to its integrated-function form to obtain the amount of substance after two hours.

\[
\frac{dA}{A^2} = k dt.
\]

This can be put into the following form,

\[
\frac{dA}{A^2} = t^0 \times k dt, \quad t^0 \neq 0,
\]

which has the form

\[
\frac{dA}{dt} = f(t) \times g(A).
\]

Hence, this problem can be solved as a separable differential equation.
\[
\frac{dA}{dt} = f(t) \times g(A) \quad \text{has form} \quad \frac{dA}{dt} = f(t) \times g(A)
\]
\[dA = kA^2 \, dt \quad \text{separate variables}
\]
\[A^{-2} \, dA = k \, dt \quad \text{prepare to integrate}
\]
\[\int A^{-2} \, dA = \int k \, dt \quad \text{integrate both sides}
\]
\[-A^{-1} + c_1 = kt + c_2 \quad \text{A to the left}
\]
\[-\frac{1}{A} = kt + c_2 - c_1 \quad \text{combine constants}
\]
\[-\frac{1}{A} = kt + c_3 \quad \text{eliminate the fraction}
\]
\[-1 = A(kt + c_3) \quad \text{eliminate the fraction}
\]
\[-\frac{1}{kt + c_3} = A \quad \text{solve for } A
\]
\[A = -\frac{1}{kt + c_3} \quad A \text{ to the left}
\]
\[100 = \frac{1}{k \times 0 + c_3} \quad \text{at } t = 0, \text{ there was 100 g}
\]
\[100 = \frac{-1}{c_3} \quad c_3 = -0.01
\]

At one hour, only 20 g of the substance remained. Substituting \(c_3 = -0.01\) into the integrated form of the function, we get

\[A = \frac{-1}{kt + c_3} \quad \text{we can solve for } k
\]
\[20 = \frac{-1}{(k \times 1) + (-0.01)} \quad \text{substitution}
\]
\[20 = \frac{-1}{k - 0.01} \quad 20(k - 0.01) = -1 \quad \text{eliminate the fraction}
\]
\[20k - 0.20 = -1 \quad \text{distributive property}
\]
\[20k = -0.8 \quad \text{isolate the variable term}
\]
\[k = -0.04 \quad \text{divide both sides by 20}
\]
\[A = \frac{-1}{-0.04t + (-0.01)} \quad \text{substitute for } k
\]
\[= \frac{-1}{-0.04t + (-0.01)} \times \frac{-100}{-100} \quad \text{simplify}
\]
\[= \frac{100}{4t + 1} \quad \text{final integrated form}
\]

How much of substance A is present after two hours?

\[A = \frac{100}{4t + 1} = \frac{100}{4 \times 2 + 1} = 11.11 \text{ g}
\]

How long before the amount of the substance decreases to 1 g?

\[1 = \frac{100}{4t + 1}
\]
\[4t + 1 = 100
\]
\[4t = 99
\]
\[t = 24.75 \text{ h}
\]

before the substance is reduced to one gram.
Differential Equations in Meteorology—Barometric Pressure

It seems counterintuitive, but air has weight. A column of air above your head is pushing down on you right now. At sea level the weight of that column of air would be more than if you were standing on top of Mt. Everest because there would be a shorter column of air over your head on top of Mt. Everest than there would be at sea level. For points on the earth’s surface, a simplistic model of barometric pressure, \( p \) (in inches of mercury in a barometer), is that, with increasing altitude, pressure decreases in direct proportion to the current pressure: 

\[
\frac{dp}{dh} = -0.2p
\]

where \( p = 29.92 \) inches of mercury at sea level when \( h = 0 \) (miles). Find the barometric pressure at the top of Mt. Everest at 29,029 ft.

\[
\frac{dp}{dh} = h^0 \times -0.2p \quad \text{form: } \frac{dp}{dh} = f(h) \times g(p)
\]

\[
\frac{dp}{p} = -0.2 \, dh \quad \text{separate variables}
\]

\[
p^{-1} \, dp = -0.2 \, dh \quad \frac{1}{p} = p^{-1}
\]

\[
\int p^{-1} \, dp = \int -0.2 \, dh \quad \text{integrate both sides}
\]

\[
\ln |p| + c_1 = -0.2h + c_2 \quad \int p^{-1} = \ln |p|
\]

\[
\ln p = -0.2h + c_3 \quad \text{constants, } p > 0
\]

\[
e^{\ln p} = e^{-0.2h+c_3} \quad a = b \quad \Rightarrow \quad e^a = e^b
\]

\[
p = e^{-0.2h} \times e^{c_3} \quad b^{m+n} = b^m \times b^n
\]

\[
p = e^{-0.2h} \times c_4 \quad \text{new constant ***}
\]

\[
29.92 = c_4 \times e^{-0.2\times0} \quad p = 29.92 \text{ at sea level}
\]

\[
29.92 = c_4 \times 1 \quad e^0 = 1
\]

\[
c_4 = 29.92
\]

\[
p = 29.92 \times e^{-0.2h} \quad \text{from *** above}
\]

\[
p = 29.92e^{-0.2\times\frac{29.029}{5.280}} \quad \text{Everest’s } h \text{ in miles}
\]

\[
p = 29.92 \times e^{-0.2\times5.497916667}
\]

\[
p = 29.92 \times e^{-1.099583333}
\]

\[
p = 29.92 \times 0.3330098089
\]

\[
p = 9.96 \text{ inches of mercury}
\]

MATLAB script for the graph at right is in Appendix A.

Author’s note: \( \frac{29.029 \text{ ft}}{5.280 \text{ ft/mi}} = 5.5 \text{ miles.} \)
Mixing as a Differential Equation

After all his salt-water fish died, the owner of a large 1,000 L cylindrical fish tank has decided to replace them with freshwater fish. The owner could have just drained and cleaned the tank but instead he decided to drain the salt water from the tank at 10 L per minute while simultaneously replacing it with freshwater. The water had been kept at 3.5% (0.035) salt content meaning that in the 1,000 L tank there was 35 kg of salt.* The solution in the tank is kept mixed during the replacement process. How much salt will be in the tank after five hours? The owner has been told that the water must be less than 0.1% salt before he can put in his freshwater fish. How long will that take?

Let \( A(t) = \) amount of salt in kilograms remaining after \( t \) minutes. Then, \( A(0) = 35 \) kg, the amount of salt at time 0: 0.035 \times 1,000 \text{ L} = 35 \text{ kg}. \( A(t) = ? \), amount of salt at time \( t \). \( A(300) = \) amount of salt at time 5 h. \( \frac{dA}{dt} = \) amount of salt coming in − amount of salt exiting. \( \frac{dA}{dt} = 0 \frac{\text{kg}}{\text{min}} - \frac{A}{100 \text{ kg}} \times 10 \frac{\text{L}}{\text{min}} \). \( \frac{dA}{dt} = \frac{0 \frac{\text{kg}}{\text{min}} - \frac{A}{100 \text{ kg}} \times 10 \frac{\text{L}}{\text{min}}}{\text{min}} \) has form \( \frac{dA}{dt} = f(t) \times g(A) \).

\[
\begin{align*}
\frac{dA}{dt} &= t^0 \times -\frac{A}{100 \text{ min}} \\
\frac{1}{A} dA &= -0.01 \text{ dt} \\
A^{-1} dA &= -0.01 \text{ dt} \\
\int A^{-1} dA &= \int -0.01 \text{ dr} \\
\ln |A| + c_1 &= -0.01 t + c_2 \\
\ln A &= -0.01 t + c_2 - c_1 \quad \text{where } A > 0 \\
\ln A &= -0.01 t + c_3 \quad \text{where } c_3 = c_2 - c_1
\end{align*}
\]

How much salt will be in the tank after five hours (300 min)? The owner has been told that the water must be less than 0.1% salt (less than one kilogram of salt in 1,000 L) for his new freshwater fish. How long will that take?

\[
\begin{align*}
A &= 35e^{-0.01xt} \quad \text{* above} \\
A &= 35e^{-0.01 \times 300} \quad 5 \text{ h} = 300 \text{ min} \\
A &= 35e^{-3} \\
A &= 1.74 \text{ g salt}
\end{align*}
\]

\[
\begin{align*}
1 &= 35 \times e^{-0.01xt} \quad \text{* above} \\
\frac{1}{35} &= e^{-0.01xt} \\
0.0285714286 &= e^{-0.01xt} \\
\ln(0.0285714286) &= \ln(e^{-0.01xt}) \\
-3.55534806 &= -0.01t \\
t &= 355.534806 \text{ min}
\end{align*}
\]

* A milliliter of water has a mass of one gram, so a liter (1,000 mL) has a mass of one kilogram. Then, 3.5% of a kilogram is 35 g, so 1,000 L must have 35 kg of salt.
In 1845 German physicist Gustav Kirchhoff (1824–1887) announced Kirchhoff’s laws, which allowed calculation of the currents, voltages, and resistances of electrical networks. Kirchhoff’s current law states that the current (flow of electrons) entering a junction is equal to the current leaving the junction. Kirchhoff’s voltage law states that, in a circuit with only resistors and inductors, the sum of the voltage drops of the resistors and the inductors in a closed loop equals the total voltage gain of the source. That is, the sum of the voltages around a closed loop is 0, \( \sum v_i = 0 \). The resistors and the inductors in a circuit both work to reduce voltage; the difference is that a resistor works to impede or reduce voltage whereas the inductor only works to impede or reduce a change in current. Kirchhoff’s voltage law is often written as \( L \frac{dI}{dt} + RI = V(t) \) using Ohm’s law \( V = IR \) as a measure of voltage drop in an inductor. Solving for \( I \) gives you an integrated formula to calculate current at a given time.

If a battery supplies a constant voltage of 30 V, the inductance is 2 H (henrys), the resistance is 10 \( \Omega \) (ohms) and \( I(0) = 0 \), find \( I(t) \) in general and the current after 0.3 seconds.

\[
\begin{align*}
L \frac{dI}{dt} + RI &= V(t) \\
2 \frac{dI}{dt} + 10I &= 30
\end{align*}
\]

\[
\frac{dI}{dt} = 15 - 5I \\
\frac{dI}{15 - 5I} = dt \\
\int \frac{dI}{15 - 5I} = \int dt
\]

Note that \( \frac{1}{5} \int u^{-1} du = -\frac{1}{5} \ln |u| \)

\[
\begin{align*}
\frac{1}{5} \ln |15 - 5I| + c_1 &= t + c_2 \\
\frac{1}{5} \ln |15 - 5I| &= t + c_2 - c_1 \\
\frac{1}{5} \ln |15 - 5I| &= t + c_3 \\
\ln |15 - 5I| &= -5t - 5c_3 \\
\ln |15 - 5I| &= -5t + c_4 \\
e^{\ln |15 - 5I|} &= e^{-5t + c_4} \\
|15 - 5I| &= e^{-5t} \times e^{c_4} \\
15 - 5I &= c_5 e^{-5t} \\
15 - 5I &= c_5 e^{-5t} \\
I &= c_6 e^{-5t} + 3 \quad **
\end{align*}
\]

MATLAB script for the graph above in Appendix A.

\[
I(0) = 0 \\
0 = c_6 e^{-5 \times 0} + 3 \\
0 = c_6 \times 1 + 3 \\
c_6 = -3
\]

Therefore, from **

\[
I = -3e^{-5t} + 3.
\]

Hence, at 0.3 seconds

\[
\begin{align*}
I &= -3e^{-5 \times 0.3} + 3 \\
I &= -3e^{-1.5} + 3 \\
I &= -3 \times 0.2231301601 + 3 \\
I &= -0.6693904804 + 3 \\
I &= 2.33060952 \text{ A at 0.3 s}
\end{align*}
\]
Newton’s Law of Cooling

Sir Isaac Newton found that the rate at which an object cools is directly proportional to the difference between the object’s temperature and the temperature of the surrounding medium (air, water). Hopefully your attention was caught by the following phrases: “rate of cooling (over time)” and “directly proportional.” Déjà vu! Something seems familiar here. Follow the following progression of thought from words to symbols.

Rate of cooling (over time) is directly proportional to the difference between the object’s temperature and the ambient temperature.

\[
\frac{dT}{dt} = k \times (T - T_a)
\]

\(T_a\) = temperature of surrounding medium (air or \(H_2O\))

\(k\) is the rate of cooling (over time) and \(T\) is object’s temperature.

\[
\int \frac{dT}{T - T_a} = k \int \, dt
\]

\[
(T - T_a)^{-1} \, dT = k \int \, dt
\]

\[
\ln|T - T_a| + c_1 = kt + c_2
\]

the \(ts\) are all confusing. Remember that the capital \(T\)s represent the object’s and the ambient temperature, whereas the lowercase \(t\) represents time.

\[
\ln|T - T_a| = kt + c_2 - c_1
\]

\[
e^{\ln(T - T_a)} = e^{kt + c_3}
\]

\(b^m \times b^n = b^{m+n}\), therefore \(e^{kt+c_3} = e^{kt}e^{c_3}\)

\(T - T_a = e^{kt}e^{c_3}\) object is cooling toward the ambient temp, therefore \(T - T_a > 0\)

\(T = T_a + e^{kt}c_4\)

\(T_f = T_a + Ce^{kt}***\) final temp = ambient temp + original temp \(\times\) a decay factor. What a mess!

There are lots of unknowns. We can simplify a bit by letting \(T_f\) be \(T_0\) when \(t = 0\).

\(T_0 = T_a + Ce^{k\times0}\)

\(T_0 = T_a + Ce^{0}\)

\(T_0 = T_a + C \times 1\)

\(C = T_0 - T_a\)

\(T_f = T_a + (T_0 - T_a) \times e^{kt}\) substitute \(C = T_0 - T_a\) into the equation above ***

A person was murdered in a mansion. Police forensic specialists arrive at (arrival time, \(t_0\)) and note the temperature of the body to be (initial temperature upon arrival, \(T_0\)). \(t\) hours later, the temperature of the body was rechecked and found to be (temperature \(t\) hours later, \(T_f\)). The thermostat in the room had been set to maintain a constant temperature of (thermostat temperature, \(T_a\)). Assume the person’s temperature was 98.6 °F while living. What was the time of death, \(t\)? Since the room was cooler than the body at the time of death, the ambient air would have acted to cool the body. Since “e” is a known constant and variables \(T_f\), \(T_a\), \(T_0\), and \(t\) are all known, you could substitute all that known information into the Newton’s Law of Cooling \(T_f = T_a + (T_0 - T_a) \times e^{kt}\), and solve for \(k\), the constant of variation for that equation.

Let’s do that again. A person was murdered in a mansion. Police forensic specialists arrive at 10 am and note the temperature of the body to be 83 °F (\(T_0\)). After 2.5 (\(t\) hours) (12:30 pm), the temperature of the body was rechecked and found to be 72 °F (\(T_f\)). The thermostat in the room had been set to maintain a constant temperature of 67 °F \((T_a)\). Assume the person’s temperature was 98.6 °F when alive. Estimate the time of death. In Newton’s Law, there are five unknowns. After substituting in for \(T_f\), \(T_a\), \(T_0\), and \(t\), we only have one unknown, \(k\), for which we can solve.
With \( k \) now known, we can rework the problem with the newly found \( k \) in conjunction with a different \( T_f \), 98.6 °F, to solve for \( t \).

\[ T_f = T_a + (T_0 - T_a) \times e^{kt} \]
\[ 72 = 67 + (83 - 67) \times e^{k \times 2.5} \]
\[ 5 = 16 \times e^{k \times 2.5} \]
\[ \frac{5}{16} = e^{k \times 2.5} \]
\[ \ln 0.3125 = \ln e^{2.5k} \]
\[ -1.16315081 = 2.5k \]
\[ k = -0.4652603239 \]

Moving to the right, the red line appears asymptotic to 67 °F. Why? Do you recall a sink asymptote from Chapter 4?

To determine the time (\( t \)) of death, substitute 98.6 °F into \( T_f \) (final temperature) and solve for \( t \).

\[ 98.6 = 67 + (83 - 67) \times e^{-0.4652603239 \times t} \]
\[ 31.6 = 16 \times e^{-0.4652603239 \times t} \]
\[ \frac{31.6}{16} = e^{-0.4652603239 \times t} \]
\[ \ln 1.975 = \ln e^{-0.4652603239 \times t} \]
\[ 0.6805683983 = -0.4652603239 \times t \]
\[ t = \frac{0.6805683964}{-0.4652603239} \]
\[ = -1.46 \]
\[ = -1:28 \]

10 am – 1:28 = 8:32 am time of death

MATLAB script for graph above in Appendix A.
Differential Equations for Continuous Compound Interest—Revisited

At the risk of oversimplifying, investments are categorized as equities (stocks), with various levels of risk, and bonds with lower, but “guaranteed,” fixed levels of income. In planning for retirement a retired person moved $300,000 out of an equity account into income-producing bonds. If that account returned 5% a year compounded continuously, how much money could the retiree count on to spend from the interest from his account each year? Well, in Chapter 3, we studied the formula \( p_f = p_0 \times e^{rt} \). For the conditions stated above, this would be \( p_1 = \$300,000 \times e^{0.05 \times 1} \).

\[ p = \$300,000 \times 1.051271096 = \$315,381.33 \]. Therefore, the retiree could withdraw \$15,381.33 a year without drawing down his principal. However, without a paycheck coming in anymore, the retiree needs more than \$15,381.33 a year to cover his living expenses. Given his family history, the retiree decides to plan on 20 years of additional life expectancy. In addition to the 5% annual growth of his money, could he draw down an additional \$23,000 a year and still not outlive his money at the end of that 20 years? We solved the previous continuous interest problem using the differential equation \( \frac{dp}{dt} = 0.05p \), so it is tempting to solve this problem with \( \frac{dp}{dt} = 0.05p - \$23,000 \). There is a flaw with that approach, but we will start out that way and then discuss the results.

\[
\frac{dp}{dt} = 0.05p - 23,000 \\
\frac{dp}{0.05p - 23,000} = dt \\
\int \frac{dp}{0.05p - 23,000} = \int dt \\
20 \int \frac{dp}{0.05p - 23,000} \times 0.05 dp = t + c_1 \\
20 \ln |0.05p - 23,000| + c_2 = t + c_1 \\
20 \ln |0.05p - 23,000| = t + c_1 - c_2 \\
20 \ln |0.05p - 23,000| = t + c_3 \\
\ln |0.05p - 23,000| = \frac{1}{20} (t + c_3) \\
\ln |0.05p - 23,000| = \frac{t}{20} + c_4 \\
\ln |0.05p - 23,000| = \frac{t}{20} + c_5 \\
e^{\ln |0.05p - 23,000|} = e^{\frac{1}{20} (t + c_3)} \\
|0.05p - 23,000| = e^{c_1} \times e^{\frac{t}{20}} \\
|0.05p - 23,000| = c_5 \times e^{\frac{t}{20}} \quad c_5 > 0
\]

\[0.05p - 23,000 = \pm(c_5 \times e^{\frac{t}{20}})\]

\[0.05p - 23,000 = c_6 \times e^{\frac{t}{20}} \quad c_6 > 0\]

\[100(0.05p - 23,000) = 100 \times c_6 \times e^{\frac{t}{20}}\]

\[5p - 2,300,000 = c_7 \times e^{\frac{t}{20}}\]

\[5p = 2,300,000 + c_7 \times e^{\frac{t}{20}}\]

\[p = 460,000 + \frac{c_7 \times e^{\frac{t}{20}}}{5}\]

\[p = 460,000 + c_8 \times e^{\frac{t}{20}} \quad t = 0, p = 300,000\]

\[300,000 = 460,000 + c_8 \times e^{0}\]

\[-160,000 = c_8 \times 1\]

\[c_8 = -160,000\]

Finally, substituting into *** above

\[p = 460,000 - 160,000 \times e^{\frac{t}{20}}\]

For \( t = 20 \) yrs

\[p_{20} = 460,000 - 160,000 \times e^{1}\]

\[p_{20} = 460,000 - 434,925.0926\]

\[p_{20} = 25,074.91\]

It appears that 20 years after drawing down his principal at $23,000 a year, he will still have principal remaining: $25,074.91. The flaw in that approach can best be shown by a visual. As was shown before, continuous compounding of principal results in a continuous accumulation of principal. However, the yearly drawdown of principal described in the problem above would be, for practical purposes, discrete: It happens all at once. If the yearly drawdown is greater than the amount of money earned during the preceding twelve months, then, over time, the original principal ($300,000) will be depleted.
If the yearly drawdown is less than the amount of money earned during the preceding twelve months, then, over time, the original principal ($300,000) will increase, but at a much slower rate than in the earlier continuous accumulation problem.

If you understand computer programming, you can use Euler’s method to simulate both continuous compounding of interest and continuous withdrawals (combining both) to analyze the problem above (continuous compounding of interest with yearly lump sum drawdown) to any specified tolerance allowable in your computer.

The following program was run first showing Euler’s method used with a $\Delta t$ of $10^{-7}$ year to calculate compound interest accumulating on $300,000 at 5% for 20 years. The answer, as you know now, would be $P_f = P_0e^{rt} = 300,000 \times e^{0.05 \times 20} = 300,000 \times e^{1} = 815,484.5485$. The answer obtained by the code was $815,484.57$. The numbers differ because Euler’s method gives discrete-math results whereas differential equations gives continuous-math results. The code that does the “continuous compounding” is combined with the code for “continuous withdrawal.” Here the term “continuous” reflects the $10^{-7} \Delta t$ used by Euler’s algorithm in the code. However, as pointed out above, “continuous” withdrawal is theoretically possible but not convenient in the real world. The yearly drawdown might be done in a lump sum at either the beginning or end of the year. This is done on the following page with the drawdown done at the beginning of the year. Notice that first of the year drawdown deprives the retiree of interest from $23,000 that year.
public class ContinuousGrowthOfPrincipal
{
    public static void main(String args[])
    {
        double t = 0;
        double r = 0.05;
        double p = 300000;
        double deltaT = 0.000001; // 10E-6
        double annualWithdrawal = 23000;
        boolean exactYear = true;
        System.out.println("Annual Withdrawal is " + AnnualWithdrawal);
        System.out.println("Delta t is " + deltaT);
        System.out.println("t p"); // column headers
        System.out.println("n" + (int)t +", " + p); // initial
        // remove Java comment symbol for output at right
        p -= annualWithdrawal; // starting annual drawdown
        while (t < 20)
        {
            // output for p shown top right
            p += ((r * p) * deltaT); // compound annual interest calculated
            // "continuously"
            t += deltaT;
            exactYear = (t - (int)t) < deltaT;
            if(exactYear)
            {
                System.out.println("n" + (int)t +", " + p);
                // remove Java comment symbol for output right
                p -= annualWithdrawal; // annual drawdown
            } // end if
        } // end while
    } // end main
} // end class

<table>
<thead>
<tr>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$300,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$315,381.33</td>
</tr>
<tr>
<td>2</td>
<td>$331,551.29</td>
</tr>
<tr>
<td>3</td>
<td>$348,550.27</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>18</td>
<td>$737,880.95</td>
</tr>
<tr>
<td>19</td>
<td>$775,712.92</td>
</tr>
<tr>
<td>20</td>
<td>$815,484.57</td>
</tr>
</tbody>
</table>

Annual withdrawal is $23,000

deltaT is 0.000001

(0, 300,000)
Since beginning algebra, you should have been encouraged to check your work by substituting the scalar answer back into the original algebraic equation.

Up to this point, all of our work has been checked using slope fields drawn in MATLAB and fitting a particular solution through given initial condition. The old way (before Mathematica and MATLAB) to check work done solving a differential equation was to take the derivative of the solution and see if it matches with the given differential equation. For the sake of nostalgia, that way to check work on a differential equation was to take the derivative of the solution and see if it matches with the given differential equation. The book will have a function that it says it got somewhere and, by substitution, check by back substituting

\[
\frac{2}{3} x - 2 = 4 \\
3 \left( \frac{2}{3} x - 2 \right) = 3 \times 4 \\
2x - 6 = 12 \\
2x = 18 \\
x = 9
\]

Will the point \((3, -1)\) lie on that particular curve when \(C = 5\)? \(3^2 + (-1)^2 \neq 5 \times (-1) \rightarrow 9 + 1 \neq -5 \rightarrow 10 \neq -5\). No.

Chapter 5 Review

Following up on the three examples of separable differential equations in Chapter 4, several more examples of that type or category of differential equation were presented. The differential equations applied to

1. chemistry
2. meteorology
3. mixing problems
4. electrical circuits
5. Newton’s Law of Cooling
6. continuous compound interest with withdrawals

Finishing out Chapter 5, an example of the pre-Mathematica, pre-MATLAB technique of checking solutions to differential equations was shown. Today many differential equation books start out showing that technique when introducing the reader to what a differential equation is. The book will have a function that it says it got somewhere and, by substitution, show that it satisfies a differential equation. The author then assumes that he/she has properly introduced what a differential equation is. Of course many beginning students will be fixated on where that “mystery” function came from and not be able to focus on the successful substitution and on the point that the author is trying to introduce.

Many, many math teachers and authors believe that math instruction should at all times be complete and rigorous and deal with all contingencies at the time of introduction. My philosophy is that sort of thinking discounts the importance of “student readiness” and is counterproductive for most beginners. What follows is a primitive chart to help beginning students (the target audience for this book) differentiate and mentally organize their thoughts to this point in the book. Bear in mind that it will have to be adjusted as more knowledge and information is covered.
Ordinary Differential Equations (ODE) by category

- Euler’s method (** Chapter 1)
- Runge–Kutta

- Only a decimal approximation can be found
- Closed form (analytical solution) and decimal approximation can be found

** Separable (** Chapters 1–5)

\[ y' = f(x) \times g(y), \ g(y) \neq 0 \]

\[ \frac{dy}{dx} = f(x) \times g(y), \ g(y) \neq 0 \]

- Exponential growth
- Exponential decay

Online Application

Visit demonstrations.wolfram.com/MixingSaltInWaterInOneTank for a great little application that will demonstrate some of the concepts from this chapter. “Mixing Salt in Water in One Tank” from the Wolfram Demonstrations Project. Contributed by Stephen Wilkerson.
Chapter 6
Solving Separable Logistic Equations

Exponential Growth versus Logistic Growth

The term *exponential growth* is often used when talking about population growth. Assuming unlimited supplies of various resources, the growth rate is proportional to the existing population. In logistic growth, the growth rate is also proportional to the existing population, but the concept takes into account limited resources, disease, and predation. Consequently, logistic growth is bounded. That boundary is called the “carrying capacity of the environment.” See the figures below.

In exponential growth, there is no upper limit, but growth is limited in logistic growth. Logistic growth is usually considered more realistic than exponential growth when studying populations. In 1798 a British economist, Thomas Malthus, became quite famous for writing that, while population growth might be exponential, shortages in food supply, war, disease and starvation create conditions which would slow that exponential growth. See the figures above.

When we studied continuous compound interest the exponential growth model was appropriate, but the logistic growth model is more appropriate when studying population growth. A better match between the problem being studied and the model used to study it generally gives more reliable study results.

Thomas Malthus
The $K$ or carrying capacity in the logistic differential equation sets a maximum bound on $P$.

$$\frac{dP}{dt} = kP$$

exponential growth

$k = 0.05$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

logistic growth

$k = 0.05, K = 2,000$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P$</th>
<th>$\text{deltaP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,050.00</td>
<td>50.00</td>
</tr>
<tr>
<td>2</td>
<td>1,102.50</td>
<td>52.50</td>
</tr>
<tr>
<td>3</td>
<td>1,157.63</td>
<td>55.13</td>
</tr>
<tr>
<td>4</td>
<td>1,215.51</td>
<td>57.88</td>
</tr>
<tr>
<td>5</td>
<td>1,276.28</td>
<td>60.78</td>
</tr>
<tr>
<td>6</td>
<td>1,340.10</td>
<td>63.81</td>
</tr>
<tr>
<td>7</td>
<td>1,407.10</td>
<td>67.00</td>
</tr>
</tbody>
</table>

... ... ...

195  13,549,189.55  645,199.50
196  14,226,649.03  677,459.48
197  14,937,981.48  711,332.45
198  15,684,880.56  746,899.07
199  16,469,124.59  784,244.03
200  17,292,580.82  823,456.23

Growth here is unbounded

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P$</th>
<th>$\text{deltaP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,025.00</td>
<td>25.00</td>
</tr>
<tr>
<td>2</td>
<td>1,049.98</td>
<td>24.98</td>
</tr>
<tr>
<td>3</td>
<td>1,074.92</td>
<td>24.94</td>
</tr>
<tr>
<td>4</td>
<td>1,099.78</td>
<td>24.86</td>
</tr>
<tr>
<td>5</td>
<td>1,124.53</td>
<td>24.75</td>
</tr>
<tr>
<td>6</td>
<td>1,149.14</td>
<td>24.61</td>
</tr>
<tr>
<td>7</td>
<td>1,173.59</td>
<td>24.44</td>
</tr>
</tbody>
</table>

... ... ...

195  1,999.91  0.005
196  1,999.91  0.0046
197  1,999.92  0.0045
198  1,999.92  0.0042
199  1,999.923 0.0040
200  1,999.927 0.0038

Growth here is bounded

---

```java
public class JavaTemplate {
    public static void main(String[] args) {
        double r = 0.05; // rate of growth
        int K = 2000; // carrying capacity
        double P = 1000; // population, principal, whatever
        System.out.println(" t P deltaP" + "\n");
        for (int t = 1; t < 200; t++) {
            double oldP = P;
            P += r * P * (K - P) / K; // logistic growth
            // P += r * P; // exponential growth
            double deltaP = P - oldP;
            System.out.println(" " + t + " " + P + " " + deltaP);
        } // end loop
    } // end main
} // end class
```
Following are five examples of situations where there is a limit to how much something can grow. Each of these situations could be modeled using the logistic growth curve.

\[ \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \]

1. **Differential Equations in Disease Control**

After spring break, a college campus nurse identified 10 students with cooties. At Day 4, she identified 30 students. There are 800 students on campus. **Assuming the infection grows logistically**, how many students will be infected after five weeks? Disease spread is limited to 800 students.

2. **Differential Equations in Sociology—Measuring the Spread of Rumors**

In a high school with 2,000 students, a couple breaks up. By the end of the day, 20 people know about it. By the end of the next day, 70 people know about it. **Assuming the gossip spreads logistically**, how many days will it take for half the school to know? Rumor spread is limited to 2,000 students.

3. **Differential Equations in Biology—Measuring Population Growth**

A population of prairie dogs is estimated to be 800 prairie dogs. After four weeks there are 1,000 prairie dogs. Biologists estimate that there is only food for 1,800 prairie dogs. **Assuming the population grows logistically**, how many prairie dogs will there be after 10 weeks? After 20 weeks? The number of prairie dogs is limited to 1,800.

4. **Differential Equations in Genetics—Monitoring Inheritance of Characteristics**

A geneticist is studying a population of mice to determine how quickly a physical trait will spread into the next generation. At the start of the study \((t = 0)\), she has established that 30% of the population has the characteristic. After five generations \((t = 5)\) she finds that 85% of the population has the characteristic. **Use the logistic function** to determine the percentage of mice that will have the studied trait after eight generations. The percentage of population is limited to 100%.

5. **Differential Equations in Advertising—Monitoring Effectiveness of an Advertising Campaign**

A company making and selling widgets is engaged in a marketing campaign in a large metropolitan area of 1 million people. **Assuming logistic growth**, use the given IVP conditions to measure the effectiveness of the campaign. The maximum number of people who could hear about the product is 1 million people.

We’ll look at each of these examples in more detail later, but first we should look a little more closely at the theory behind logistic equations and their relationship to separable differential equations. Logistic equations are a subset of separable differential equations, although this is not evident when comparing the two:

\[ \frac{dp}{dt} = kP \quad \text{separable differential equation} \]

\[ \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \quad \text{logistic differential equation} \]

Hold onto this idea as you proceed through your differential-equations class. There will be many occasions where more than one technique can be used to solve a problem.
If \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right) \) is separable, then it should be possible to convert it to the form: \( f(t) \times g(P) \). Let’s try.

**Attempt #1**

\[
\begin{align*}
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) \\
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) dt \\
\frac{dP}{kP} &= \left( kP \left( 1 - \frac{P}{K} \right) \right) dt \\
\frac{dP}{kP} &= kP \left( 1 - \frac{P}{K} \right) dt \\
\text{?????????????}
\end{align*}
\]

**Attempt #2**

\[
\begin{align*}
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) \\
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) dt \\
\frac{dP}{kP} &= \left( kP \left( 1 - \frac{P}{K} \right) \right) dt \\
\frac{dP}{kP} &= k \left( 1 - \frac{P}{K} \right) dt \\
\frac{dP}{kP} &= \frac{dP}{kP} \text{ separated!}
\end{align*}
\]

**Attempt #3**

\[
\begin{align*}
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) \\
\frac{dP}{dt} &= kP \left( 1 - \frac{P}{K} \right) dt \\
\frac{dP}{kP} &= \left( kP \left( 1 - \frac{P}{K} \right) \right) dt \\
\frac{dP}{kP} &= kP \left( 1 - \frac{P}{K} \right) dt \\
\text{?????????????}
\end{align*}
\]

The integration of the left side in #3 can be very difficult. It turns out that there is a way to avoid such difficult integrations. Chapter 6 teaches a clever way to do this. Be patient. What follows is a review from beginning algebra.

It is often helpful to bridge into new ideas by connecting them to old ones.

**Solve the following quadratic equation.**

\[
x^2 - 2x - 15 = 0 \quad \text{original quadratic}
\]

\[
(x + 3)(x - 5) = 0 \quad \text{factor the trinomial into the product of two binomials}
\]

\[
x + 3 = 0 \text{ or } x - 5 = 0 \quad \text{zero product property, if } a \times b = 0 \text{ then } a = 0, b = 0, \text{ or both.}
\]

\[
x = -3 \text{ or } x = 5
\]

But wait! What could you do if the quadratic did not factor? In that case you switched to another attack called “Completing the square.”

Solve

\[
2x^2 - 9x - 4 = 0 \quad \text{original quadratic}
\]

\[
x^2 - \frac{9}{2}x - \frac{4}{2} = 0 \quad \text{divide both sides by 2 so that the } x^2 \text{ term has a coefficient of 1}
\]

\[
x^2 - \frac{9}{2}x = \frac{4}{2} \quad \text{isolate the } x \text{ terms}
\]

\[
x^2 - \frac{9}{2}x + \frac{81}{16} = 2 + \frac{81}{16} \quad \text{complete the binomial square by adding } \left( \frac{1}{2} \times \frac{9}{2} \right)^2 \text{ to both sides}
\]

\[
\left( x - \frac{9}{4} \right)^2 = \frac{32}{16} + \frac{81}{16} \quad \text{change form: } a^2 - 2ab + b^2 = (a - b)^2, \text{ prepare to add fractions}
\]

\[
\left( x - \frac{9}{4} \right)^2 = \frac{113}{16} \quad \text{add like fractions}
\]

\[
x - \frac{9}{4} = \pm \frac{\sqrt{113}}{4} \quad \text{take the square root of both sides}
\]

\[
x = \frac{9}{4} \pm \frac{\sqrt{113}}{4} \quad \text{isolate the } x
\]

\[
x = \frac{9 \pm \sqrt{113}}{4} \quad \text{combine like fractions}
\]

Oh, that was fun. Let’s do it again! Solve the following quadratic equation.

\[
3x^2 - 7x - 5 = 0
\]

Just kidding!! We could solve this equation using the technique shown above. However, the process for solving this equation will be the same as the process for solving the first one above with only the coefficients being different. Perhaps we could do the same process with symbols \((a, b, \text{ and } c)\) for the constants, \(ax^2 + bx + c = 0\); then we would have a generic template for all quadratics.


The same sort of thing happens when working with “logistic equations.” The differential equation form of a “logistic equation” is $\frac{dP}{dt} = kP(1 - \frac{P}{K})$ where $P$ is the population at time $t$, $k$ is the constant of proportionality, and $K$ is the “carrying capacity.” From that point forward there is a significant amount of calculus and algebraic acrobatics needed to solve this equation for the specifics of the equation at hand, very much like all that effort shown above to solve an unfactorable quadratic equation by the process of completing the square. Just like finding the general-form quadratic equation, $ax^2 + bx + c = 0$, once to get a general-form answer, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it is best to take the time to solve the general-form logistic differential equation, $\frac{dP}{dt} = kP(1 - \frac{P}{K})$, for its analytic form, $P(t) = \frac{K}{Ae^{-kt} + 1}$, where $A = \frac{K - P_0}{P_0}$. From that point forward, we can substitute into the problem-specific analytic (integrated) form quickly, and relatively painlessly obtain the desired information.

If you wish to know how to convert from the general logistic differential equation form to its algebraic (differentiated) form please do access Appendix B. It is two pages long and rather involved.
Now let’s return to those concrete examples! Following are five examples of situations where there is a limit on how much something can grow: cooties, a rumor, genetic transfer, population growth, and product awareness. Therefore, each of these situations could be modeled using the logistic growth curve, \( \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \), or its equivalent integrated form.

<table>
<thead>
<tr>
<th>Differential Equation Using the Logistic Model</th>
<th>Integrated-Equation Form Using the Logistic Model</th>
</tr>
</thead>
</table>
| \( \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \) … this is separable, but how???
| \( P = \frac{K}{(A \times e^{-kt} + 1)} \), where \( A = \frac{K - P_0}{P_0} \) (See Appendix B for a derivation of this formula.) |

**Ex 1. Differential Equations in Disease Control—Logistic Model**

After spring break, a college campus nurse identified 10 students with cooties. At Day 4, she identified 30 infected students. There are 800 students on campus. Assuming the logistic model, how many students will be infected after two weeks? After five weeks?

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \quad \text{carrying capacity} = 800
\]

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{800}\right) \quad \text{switch to integrated form}
\]

\[
P = \frac{K}{(A \times e^{-kt} + 1)} \quad \text{where} \quad A = \frac{K - P_0}{P_0}
\]

**Step 1:** Identify the carrying capacity and the initial population. Solve for \( A \).

\[
A = \frac{K - P_0}{P_0} = \frac{800 - 10}{10} = \frac{790}{10} = 79
\]

**Step 2:** Use \( K \) and \( A \) to solve for \( k \) on Day 4 using the integrated-equation form

\[
P = \frac{K}{(A \times e^{-kt} + 1)}
\]

\[
30 = \frac{800}{(79 \times e^{-4k} + 1)}
\]

\[
3 = \frac{80}{(79 \times e^{-4k} + 1)}
\]

\[
3 \times (79 \times e^{-4k} + 1) = 80
\]

\[
(79 \times e^{-4k} + 1) = \frac{80}{3} \quad \text{divide both sides by 3}
\]

\[
79 \times e^{-4k} = \frac{80}{3} - \frac{3}{3} = \frac{77}{3}
\]

\[
79 \times e^{-4k} = \frac{77}{3} \quad \text{subtract 1 from both sides}
\]

\[
e^{-4k} = \frac{77}{3} \times \frac{1}{79} = \frac{77}{237}
\]

\[
e^{-4k} = 0.3248945148 \quad \text{convert} \frac{77}{237} \text{to a decimal}
\]

\[
\ln(e^{-4k}) = \ln(0.3248945148) \quad \text{take the natural log of both sides}
\]

\[-4k = -1.124254719 \quad \ln e^u = u \]

\[k = 0.2810636798 \quad \text{finally, solve for} \ k\]
Step 3: Use $A$, $K$ and $k$ to determine logistic population numbers after two and five weeks.

$$P = \frac{K}{(A \times e^{-kt} + 1)}$$

After 2 weeks (14 days)

$$P = \frac{800}{79 \times e^{-0.2810636798} + 1}$$
$$P = \frac{800}{79 \times e^{-3.934891517} + 1}$$
$$P = \frac{800}{1.544277766 + 1}$$
$$P = \frac{800}{2.544277766}$$

$P = 314$ infected

After 5 weeks (35 days)

$$P = \frac{800}{79 \times e^{-0.2810636798 \times 35} + 1}$$
$$P = \frac{800}{79 \times e^{-9.837228793} + 1}$$
$$P = \frac{800}{0.004220588 + 1}$$
$$P = \frac{800}{1.0042205875}$$

$P = 797$ infected

MATLAB script for graph below in Appendix A.
Ex 2. Differential Equations in Sociology—Measuring the Spread of Rumors

In a high school with 2,000 students, a couple breaks up. By the end of the day, 20 people know. By the end of the next day, 70 people know. If the logistic model holds, how many days will it take for before half the school knows?

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \quad \text{carrying capacity} = 2,000
\]

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{2,000}\right) \quad \text{switch to integrated form}
\]

\[
P = \frac{K}{(A \times e^{-kt} + 1)} \quad \text{where } A = \frac{K - P_0}{P_0} \text{ and } P_0 = 20
\]

Step 1: Identify the carrying capacity and the initial population.

Solve for \(A\).

\[
A = \frac{K - P_0}{P_0} = \frac{2,000 - 20}{20} = \frac{1,980}{20} = 99
\]

Step 2: Use \(K\) and \(A\) to solve for \(k\) on Day 1 using the integrated-equation form (70 people know at the end of Day 1).

\[
P = \frac{K}{(A \times e^{-kt} + 1)}
\]

\[
70 = \frac{2,000}{(99 \times e^{-k \times 1} + 1)}
\]

multiply to eliminate the fractions

\[
70 \times \left(99 \times e^{-k} + 1\right) = 2,000
\]

divide both sides by 70

\[
99 \times e^{-k} = 27.57142857
\]

subtract 1 from both sides

\[
e^{-k} = 0.2784992785
\]

divide away the 99

\[
\ln(e^{-k}) = \ln(0.2784992785) \quad \text{take the natural log of both sides}
\]

\[
-k = -1.27833981
\]

\[
\ln e^u = u \text{ on left, ln value taken on right}
\]

\[
k = 1.27833981
\]

Step 3: Use \(A\), \(K\) and \(k\) to determine how many days before half the school knows.

\[
P = \frac{K}{(A \times e^{-kt} + 1)} \quad \text{integrated form}
\]

\[
1,000 = \frac{2,000}{(99 \times e^{-1.27833981t} + 1)}
\]

substitute \(\frac{1}{2} \times 2,000 = 1,000\) students into the integrated logistic growth model

\[
\frac{1}{2} = \frac{1}{99 \times e^{-1.27833981t} + 1}
\]

divide both sides by 2,000

\[
(99 \times e^{-1.27833981t} + 1) \times 1 = 2 \times 1
\]

cross multiply the proportion

\[
99 \times e^{-1.27833981t} + 1 = 2 \quad a \times 1 = a
\]

\[
99 \times e^{-1.27833981t} = 1
\]

subtract 1 from both sides

\[
e^{-1.27833981t} = \frac{1}{99}
\]

divide away the 99

\[
e^{-1.27833981t} = 0.01010101 \quad \text{change } \frac{1}{99} \text{ to a decimal}
\]

\[
\ln(e^{-1.27833981t}) = \ln(0.010101) \quad a = b \text{ iff } \ln a = \ln b
\]

\[
-1.27833981t = -4.59511985 \quad \ln e^u = u
\]

\[
t = 3.59 \text{ days}
\]
Ex 3. Differential Equations in Game Management—Measuring Population Growth

A population of prairie dogs is estimated to be 800. After four weeks, there are 1,000 prairie dogs. Biologists estimate that there is only food for 1,800 prairie dogs. Assuming the logistic model holds, how many prairie dogs there will be after 10 weeks? 20 weeks? 30 weeks?

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right)
\]

where \( k \) is the growth rate and \( K \) is the carrying capacity.

\[ P(t) = \frac{K}{\left(A \times e^{-kt} + 1\right)} \]

where \( A = \frac{K - P_0}{P_0} \), integrated form of the logistic-model equation

Step 1: Solve for \( A \).

\[
A = \frac{1,800 - 800}{800} = \frac{1,000}{800} = 1.25
\]

\[
P = \frac{1,800}{\left(1.25 \times e^{-k\times4} + 1\right)}
\]

Step 2: Use \( K \) and \( A \) to solve for \( k \) at time four weeks

\[
P = \frac{K}{\left(A \times e^{-kt} + 1\right)}
\]

1,000 = \[
\frac{1,800}{\left(1.25 \times e^{-k\times4} + 1\right)}
\]

substitute for \( A \), \( K \), and \( t \) into the integrated form of logistic-model equation

1,000 \times (1.25 \times e^{-k\times4} + 1) = 1,800 multiply to eliminate the fraction

(1.25 \times e^{-k\times4} + 1) = 1.8 divide away the 1,000
Differential Equations: A Visual Introduction for Beginners

$$1.25 \times e^{-k \times 4} = 0.8$$ subtract 1 from both sides
$$e^{-k \times 4} = \frac{0.8}{1.25}$$ divide away the 1.25
$$e^{-k \times 4} = 0.64$$ convert fraction to a decimal
$$\ln e^{-k \times 4} = \ln 0.64 \quad a = b \text{ iff } \ln a = \ln b$$
$$-k \times 4 = -0.4462871026$$ \quad \ln e^u = u \text{ on left, ln value on right}
$$k = 0.1115717757$$ solve for \(k\)

Substitute \(A\), \(K\), and \(k\) into the integrated logistic-growth-model equation. Solve for \(t = 10, 20, 30\).

\[
P = \frac{K}{(A \times e^{-kt} + 1)}, \quad \text{for } t = 10
\]
\[
P = \frac{1,800}{(1.25 \times e^{-0.1115717757 \times 10} + 1)}
\]
\[
P = \frac{1,800}{(1.25 \times e^{-1.1115717757} + 1)} \quad \text{multiply by 10 in the exponent of } e
\]
\[
P = \frac{1,800}{(1.25 \times 0.32768000 + 1)} \quad \text{calculate } e^{-1.1115717757}
\]
\[
P = \frac{1,800}{(0.4096000 + 1)} \quad \text{calculate product}
\]
\[
P = \frac{1,800}{1.4096000} \quad \text{add 1 in the denominator}
\]
\[
P = 1,277
\]

\[
P = \frac{K}{(A \times e^{-kt} + 1)}, \quad \text{for } t = 20
\]
\[
P = \frac{1,800}{(1.25 \times e^{-0.1115717757 \times 20} + 1)}
\]
\[
P = \frac{1,800}{(1.25 \times e^{-2.231435514} + 1)} \quad e^{-0.1115717757 \times 20} = e^{-2.231435514}
\]
\[
P = \frac{1,800}{(1.25 \times 0.1073741823 + 1)} \quad \text{calculate } e^{-2.231435514}
\]
\[
P = \frac{1,800}{(0.1342177279 + 1)}
\]
\[
P = \frac{1,800}{1.134217728}
\]
\[
P = 1,587
\]

\[
P = \frac{K}{(A \times e^{-kt} + 1)}, \quad \text{for } t = 30
\]
\[
P = \frac{1,800}{(1.25 \times e^{-0.1115717757 \times 30} + 1)}
\]
\[
P = \frac{1,800}{(1.25 \times e^{-3.3471532710} + 1)} \quad e^{-0.1115717757 \times 30} = e^{-3.3471532710}
\]
\[
P = \frac{1,800}{(1.25 \times 0.0351843720 + 1)} \quad \text{calculate } e^{-3.3471532710}
\]
\[
P = \frac{1,800}{(0.0439804650 + 1)}
\]
\[
P = \frac{1,800}{1.0439804650}
\]
\[
P = 1,724
\]
Ex 4. Differential Equations in Genetics—Monitoring Inheritance of Characteristics

A geneticist is studying mice to determine how quickly a physical trait will spread through a population from one generation to the next. At the start of the study \((t = 0)\), she has established that 30% of the population has the characteristic. After five generations \((t = 5)\), she finds that 85% of the population has the characteristic. Assume the logistic growth model to determine the percentage of mice that will have the studied trait after eight generations and after 10 generations.

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right)
\]

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{100\%}\right)
\]

---

### Step 1: Solve for \(A\).

\[
A = \frac{K - P_0}{P_0} = \frac{100\% - 30\%}{30\%} = \frac{70\%}{30\%} = 2.333
\]

### Step 2: After five generations

\[
P = \frac{K}{A \times e^{-kt} + 1}
\]

\[
P = \frac{K}{2.333 \times e^{-5k} + 1}
\]

\[
0.85 \times \left(2.333 e^{-5k} + 1\right) = 1
\]

\[
2.333 e^{-5k} + 1 = 1.176
\]

\[
e^{-5k} = 0.176
\]

\[
\ln e^{-5k} = \ln(0.176) \quad a = b \text{ if } \ln a = \ln b
\]

\[
-5k = -2.582 \quad \ln e^a = u
\]

\[
k = 0.516
\]

---

### After eight generations

\[
P = \frac{100\%}{2.333 e^{-0.516 \times 8} + 1}
\]

\[
P = \frac{1}{2.333 e^{-4.131} + 1}
\]

\[
P = \frac{1}{2.333 \times 0.0161 + 1}
\]

\[
P = \frac{1}{0.037 + 1}
\]

\[
P = \frac{1}{1.037}
\]

\[
P = 0.9638
\]

---

### After ten generations

\[
P = \frac{100\%}{2.333 e^{-0.516 \times 10} + 1}
\]

\[
P = \frac{1}{2.333 e^{-5.164} + 1}
\]

\[
P = \frac{1}{2.333 \times 0.0057 + 1}
\]

\[
P = \frac{1}{0.013 + 1}
\]

\[
P = \frac{1}{1.013}
\]

\[
P = 0.9868
\]

---

**MATLAB script for graph above in Appendix A.**

**Table:**

<table>
<thead>
<tr>
<th>Generation</th>
<th>Have the trait</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30.0%</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>85.0%</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>96.4%</td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>98.7%</td>
</tr>
</tbody>
</table>
Ex 5. Differential Equation Modeling Advertising Awareness

A company making and selling widgets is engaged in a marketing campaign in a large metropolitan area of 1 million people. At the beginning of an advertising campaign, only 10% (100,000) of the citizens had heard of the product. At the end of one year, 40% (400,000) of the citizens had heard of the product. Assuming the amount of advertising stays the same and assuming that awareness of the product grows as in the logistic model, how many people will have heard of the product at the end of the second year? At the end of the third year?

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right)
\]

maximum awareness

\[
P = \frac{K}{A \times e^{-kt} + 1}, \quad \text{where} \quad A = \frac{K - P_0}{P_0}
\]

Step 1: Solve for \(A\).

\[
A = \frac{K - P_0}{P_0} = \frac{1,000,000 - 100,000}{100,000} = \frac{900,000}{100,000} = 9
\]

Step 2: Solve for \(k\) at Year 1

\[
400,000 = \frac{1,000,000}{9 \times e^{-k \times 1} + 1} \quad \text{substitute for} \quad A, K, t
\]

\[
4 = \frac{10}{9e^{-k \times 1} + 1}
\]

\[
4 \times (9e^{-k \times 1} + 1) = 10
\]

\[
9e^{-k} + 1 = 2.5
\]

\[
9e^{-k} = 1.5
\]

\[
e^{-k} = 0.166666667 \quad \text{divide away the} \quad 9
\]

\[
\ln(e^{-k}) = \ln(0.166666667) \quad a = b \text{ if } \ln a = \ln b
\]

\[
-k = -1.791759469
\]

\[
k = 1.791759469
\]

Substitute to find awareness after Year 2

\[
P = \frac{1,000,000}{9e^{-1.791759473 \times 2} + 1}
\]

\[
P = \frac{1,000,000}{9e^{-3.583518938} + 1}
\]

\[
P = \frac{1,000,000}{9 \times 0.0277777778 + 1}
\]

\[
P = \frac{0.250000000 + 1}{1,000,000}
\]

\[
P = \frac{1,25000000}{1.041666666}
\]

\[
P = 800,000 \text{ people had heard of the product}
\]

Substitute to find awareness after Year 3

\[
P = \frac{1,000,000}{9e^{-1.791759469 \times 3} + 1}
\]

\[
P = \frac{1,000,000}{9e^{-5.375278408} + 1}
\]

\[
P = \frac{9 \times 0.0046296296 + 1}{1,000,000}
\]

\[
P = \frac{0.0416666667 + 1}{1.041666666}
\]

\[
P = 960,000 \text{ people had heard of the product}
\]
A better match between the problem being studied and the model used to study it generally provides more reliable study results. The logistic-growth model is only one of several hypothetical models that can be used to study phenomena. Following are a few others that we will not cover in this book.
Chapter 6 introduced a new kind of differential equation, a logistic differential equation. The exponential-growth differential equation differs from logistic differential equation in that the latter had some sort of natural bounds placed upon the upper limit of growth of the data being observed. Five examples were given:

1. disease control
2. sociology
3. population growth bounded by available food
4. the spread of genetic traits
5. advertising awareness

Although logistic equations were a type of separable differential equation, it was noted that their signature form and resulting curve differed.
It was observed that the algebra necessary to solve a logistic differential equation was significantly more difficult than solving a separable exponential-growth problem, and considerable time, effort, and page space (Appendix B) were spent on obtaining a closed algebraic formula

\[ P = \frac{K}{(A \times e^{-kt} + 1)} \]

where \( A = \frac{K - P_0}{P_0} \)

which would allow for a “plug-and-chug substitution solution” for all future logistic differential equations. Thus, all the tedious details involved in the development of the generic form were eliminated from all future logistic-problem situations.

Several other modeling curves were shown for the reader’s consideration, and it was noted that a model chosen to more closely model a situation generally produces better prediction results.

It was pointed out that logistic differential equations are a subset of separable differential equations, so it should be obvious that categorizing the different types of differential equations will become more complicated. The following chart should be considered a primitive and simplistic attempt to help beginning students (the target audience for this book) review, differentiate, and mentally organize their thoughts to this point in the book.

---

**Online Application**

Visit demonstrations.wolfram.com/LogisticEquation for a great little application that will demonstrate some of the concepts from this chapter. “Logistic Equation” from the Wolfram Demonstrations Project. Contributed by Jeff Bryant.
Appendix B

Converting a Logistic Differential Equation into Its Algebraic (Integrated) Equivalent

Chapter 6 introduces the idea of a *logistic differential equation*. Since logistic differential equations—equations of the form \( \frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \)—are a subset of separable differential equations—equations of the form \( \frac{dv}{dx} = f(x) \times g(y), g(y) \neq 0 \)—they can be separated and solved as separable equations.

However, because of the complications involved in making that transformation, it is considered best to convert a "generic logistic equation" into its *generic algebraic (integrated) form* and just use the algebraic form whenever the need arises to solve a logistic differential equation. Remember that Chapter 6 makes the analogy with the idea of solving a general quadratic equation, \( ax^2 + bx + c = 0 \), using the quadratic formula, \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

The following text shows how to convert from the generic Differential Equation for a Logistic Model to its algebraic or integrated form. It is located here in Appendix B for those students who are curious as to how this conversion is done. It is not really necessary for application-oriented students to understand the following logic in order to solve logistic differential equations, but it is definitely desirable!! That is the difference between “mathmagic” and mathematics.

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \quad \text{original logistic differential equation form}
\]

\[
dP = kP \left(1 - \frac{P}{K}\right) \, dt \quad \text{separation of differential } dP \text{ from differential } dt
\]

\[
dP = kP \left(\frac{K}{K} - \frac{P}{K}\right) \, dt \quad \text{preparing to combine fractions}
\]

\[
dP = kP \left(\frac{K}{K} - \frac{P}{K}\right) \, dt \quad \text{combine like fractions}
\]

\[
K \, dP = kP (K - P) \, dt \quad \text{multiplying by } K \text{ ("carrying capacity") on both sides}
\]

\[
\frac{K}{P(K - P)} \, dP = k \, dt \quad \text{grouping variables } P \text{ and differential } dP \text{ to the left}
\]
\[
\frac{K}{P(K - P)} \, dP = k \, dt \quad \text{recopying from above after arithmetic detour}
\]
\[
\frac{A(K - P) + BP}{P(K - P)} \, dP = k \, dt \quad \text{>>> substituting for } K \text{ from notes above} \lllllllll
\]
\[
\frac{1(K - P) + 1 \times P}{P(K - P)} \, dP = k \, dt \quad \text{substituting into } A \text{ and } B \text{ from notes above}
\]
\[
\frac{(K - P) + P}{P(K - P)} \, dP = k \, dt
\]
\[
\frac{P + (K - P)}{P(K - P)} \, dP = k \, dt \quad \text{commutative property of addition}
\]
\[
\left[ \frac{P}{P(K - P)} + \frac{(K - P)}{P(K - P)} \right] \, dP = k \, dt \quad \text{splitting the numerator}
\]
\[
\left( \frac{1}{K - P} + \frac{1}{P} \right) \, dP = k \, dt \quad \text{canceling } \frac{P}{P} \text{ and } \frac{K - P}{K - P}
\]
\[
\int \left( \frac{1}{K - P} + \frac{1}{P} \right) \, dP = \int k \, dt \quad \text{integrating both sides}
\]
\[
\int \frac{1}{K - P} \, dP + \int \frac{1}{P} \, dP = kt + c_1 \quad \text{integral of a sum on left, integral of } k \, dt \text{ on right}
\]
\[
\int (K - P)^{-1} \, dP + \int P^{-1} \, dP = kt + c_1
\]
\[
- \int (K - P)^{-1}(-1) \, dP + \int P^{-1} \, dP = kt + c_1 \quad \text{preparing to apply } \int u^{-1} \, du = \ln u
\]
\[
-ln |K - P| + c_2 + \ln |P| + c_3 = kt + c_1 \quad \int u^{-1} \, du = \ln u \text{ applied twice on left}
\]
\[
-ln |K - P| + \ln |P| = kt + c_4 \quad \text{combining all three constants of integration}
\]
\[
-1 \times [-\ln |K - P| + \ln |P|] = -(kt + c_4) \quad \text{eliminating the leading -1 from the left side}
\]
\[
\ln |K - P| - \ln |P| = -kt - c_4 \quad \text{distributive property both sides}
\]
\[
\ln \left| \frac{K - P}{P} \right| = -(kt + c_4) \quad \text{if } m = n, \text{ then } b^m = b^n
\]
\[
e^{\ln \left| \frac{K - P}{P} \right|} = e^{-kt - c_4} \quad e^{\ln u} = u \text{ on left, } e^{a+b} = e^a \times e^b \text{ on right}
\]
\[
\left| \frac{K - P}{P} \right| = c_5 \times e^{-kt} \quad \text{substituting } e^{-c_4} = c_5
\]
\[
\left\{ \frac{K - P}{P} \right\} = \pm c_5 \times e^{-kt} \quad \text{removing the absolute value}
\]
\[
\frac{K - P}{P} = c_6 \times e^{-kt} \quad \text{substituting } \pm c_5 = c_6
\]
\[ \frac{K - P}{P} = A e^{-kt} \]

arbitrary substitution so that the final form is in the standard format.

\[ \frac{K}{P} - 1 = A \times e^{-kt} \]

splitting the numerator: \( \frac{K}{P} - \frac{P}{P} = \frac{K}{P} - 1 \)

\[
\begin{align*}
\frac{K}{P} &= A \times e^{-kt} + 1 \\
K &= (A \times e^{-kt} + 1) \times P
\end{align*}
\]

\[ \frac{K}{(A \times e^{-kt} + 1)} = P \]

\[ P = \frac{K}{(A \times e^{-kt} + 1)} \]

where \( A = \frac{K - P_0}{P_0} \)

Whew! Thanks patrickjmt on YouTube!!